PICARD AND TAYLOR KERNELS FOR SELF-ADJOINT SECOND ORDER DIFFERENTIAL EQUATIONS

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AN ABSTRACT OF A THESIS

PICARD AND TAYLOR KERNELS FOR SELF-ADJOINT SECOND ORDER DIFFERENTIAL EQUATIONS

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Doctor of Arts in Mathematics

This thesis examines Gronwall's inequality, its generalizations, and its applications. The research aims to expand on the ideas of Gronwall's inequality and reveal further generalizations. The previous research on Picard kernels provides inspiration for the development of Taylor kernels, which are extensively explored in terms of integral equations. The integral equation findings are then completely translated to the differential equation setting. The main results are used to give a new proof as well as an integral version of Sturm's Comparison Theorem. The final result concerns the oscillation of solutions to a second-order differential equation.

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CHAPTER 1
INTRODUCTION

Differential and integral inequalities have long been used as tools for studying differential and integral equations. Much research has been dedicated to such inequalities because of the vast number of applications to dynamical systems in the natural and technological world, and many of these dynamical problems give rise to models which involve differential or integral equations, few of which have accessible solutions. Inequalities that give definite bounds on these solutions provide a useful tool in determining properties of solutions to these dynamical systems. Perhaps one of the most used inequalities for this purpose is *Gronwall's inequality.*

Gronwall's inequality provides us with a way to bound functions which satisfy a simple integral equation. In application, this produces many useful estimates when considering first-order differential equations. This thesis aims to generalize this basic idea to higher-order ordinary differential equations and apply these results to explore when a Picard-like iteration technique can be effectively applied in order to produce estimates for solutions. In this section, we will look at the classical version of Gronwall's inequality, provide a slightly different formulation of it which will be useful for the results in the following chapters, provide a survey of generalizations and results that follow directly from Gronwall's inequality, and discuss some applications.

1.1 Gronwall's Inequality

In 1919, Thomas Hakon Gronwall proved a version of the following theorem in which B and K were constant terms [1]. In 1943, Richard Bellman proved the generalized result, still referred to as Gronwall's inequality, as it is stated below [2].

For this reason, Theorem 1.1.1 is often referred to as the Bellman-Gronwall Inequality.

Theorem 1.1.1 (Gronwall's inequality)**.** *Let* $B, K \in C[a,b]$ *with* $K \geq 0$. *If* $u \in C[a,b]$ *satisfies*

$$u(t) \leq B(t) + \int_a^t K(s)u(s)ds, \quad t \in [a,b],$$

then

$$u(t) \leq B(t) + \int_a^t K(s)B(s)\exp\left[\int_s^t K(u)\,du\right] ds$$

in $[a,b]$.

Proof. Define

$$y := \int_a^t K(v)u(v)dv, \ t \in [a,b].$$

Then $y(a) = 0$ and, by the Fundamental Theorem of Calculus,

$$y'(t) = K(t)u(t) \leq K(t)B(t) + K(t)\int_a^t K(s)u(s)ds = K(t)B(t) + K(t)y(t),$$

$t \in (a,b)$. Multiply y by $F(t) = \exp\left[-\int_a^t K(s)\,ds\right]$. Then

$$\begin{aligned}
(y(t)F(t))' &= y'(t)F(t) + y(t)F'(t) \\
&\le (K(t)B(t) + K(t)y(t))F(t) \;-\; y(t)F(t)K(t) \\
&= K(t)B(t)F(t)\,.
\end{aligned}$$

Integrate:

$$\int_a^t \frac{d}{du}(y(u)F(u))\,du = y(t)F(t)$$

so that

$$y(t)F(t) \le \int_a^t K(u)B(u)F(u)\,du.$$

Multiply the inequality by $1/F(t) > 0$:

$$\begin{aligned}
y(t) &\le \frac{1}{F(t)}\int_a^t K(u)B(u)F(u)\,du \\
&= \int_a^t K(u)B(u)\exp\left[\int_u^t K(s)\,ds\right]du\,.
\end{aligned}$$

By hypothesis, $u(t) \le B(t) + y(t)$. Therefore

$$u(t) \le B(t) + \int_a^t K(s)B(s)\exp\left[\int_s^t K(u)\,du\right]ds.$$

■

For the purposes of this thesis it is useful to present a slightly reformulated version of Gronwall's inequality, which will provide the general setting for many of the results. We restrict the interval to $[0, b]$ for ease of argument, but note that all results in this thesis are valid for the more general interval $[a, b]$.

Theorem 1.1.2. *Let $B, K \in C[0, b]$ with $K \geq 0$. If $u \in C[0, b]$ satisfies*

$$u(t) \leq B(t) + \int_0^t K(s)u(s)ds, \quad t \in [0, b],$$

and $\phi \in C[0, b]$ is the solution of

$$\phi(t) = B(t) + \int_0^t K(s)\phi(s)ds, \quad t \in [0, b],$$

then $u \leq \phi$ on $[0, b]$.

1.2 Survey of Results and Generalizations

Gronwall's theorem is considered so remarkable and useful that many mathematicians have (and continue to) set out to generalize it and maximize its usefulness. The following results are just a few of these, presented for the purpose of giving the reader a taste of the many generalizations that have been proven. This theorem, proven by Silvestru Dragomir, is a direct result of Gronwall's inequality [4].

Theorem 1.2.1. *Let $B, K \in C[a, b]$ with $K \geq 0$ and B differentiable on $[a, b]$. If $u \in C[a, b]$ satisfies*

$$u(t) \leq B(t) + \int_a^t K(s)u(s)ds, \quad t \in [a, b],$$

then

$$u(t) \leq B(t)\left[\int_a^t K(v)dv\right] + \int_a^t \exp\left[\int_s^t K(v)dv\right] B'(s)ds$$

in $[a,b]$.

H.E. Gollwitzer gave the following two generalizations [5]. In the first theorem, Gollwitzer shows that you can multipy the integral in Gronwall's inequality by a nonnegative, continuous function $g(t)$. Gronwall's inequality is a special case of this situation with $g(t) = 1$.

Theorem 1.2.2. *Let* u, B, g, *and* K *be nonnegative, continuous functions defined on* $J = [a,b]$, *and let* u *satisfy*

$$u(t) \leq B(t) + g(t)\int_a^t K(s)u(s)\,ds, t \in J.$$

Then on the interval J

$$u(t) \leq B(t) + g(t)\int_a^t K(s)B(s)\exp\left[\int_s^t K(v)g(v)\,dv\right] ds.$$

Theorem 1.2.3. *Let* u, v, K, *and* g *be nonnegative, continuous functions defined on* $J = [a,b]$ *and*

$$u(t) \geq v(t) - g(t)\int_x^t K(s)v(s)\,ds,\ a \leq x \leq t \leq b.$$

Then

$$u(t) \geq v(x)\exp\left[-g(t)\int_x^t K(s)\,ds\right],\ a \leq x \leq t \leq b.$$

The following two generalizations were given by Deepak B. Pachpatte [3].

Theorem 1.2.4. *Let u, g, and k be nonnegative, continuous functions defined on $J = [a, b]$, $B(t)$ be a continuous, positive, and nondecreasing function defined on J, and let u satisfy*

$$u(t) \leq B(t) + g(t) \int_a^t k(s)u(s)\, ds, t \in J.$$

Then on the interval J

$$u(t) \leq B(t) \left[1 + g(t) \int_a^t k(s) \exp\left[\int_s^t k(v)g(v)\, dv\right] ds\right].$$

Theorem 1.2.5. *Let u, B, g, K, and q be nonnegative, continuous functions defined on $J = [a, b]$ and*

$$u(t) \leq B(t) + g(t) \int_a^t [K(s)u(s) + q(s)]\, ds, t \in J.$$

Then on the interval J

$$u(t) \leq B(t) + g(t) \int_a^t [K(s)B(s) + q(s)]\, exp\left[\int_s^t K(v)q(v)\, dv\right] ds.$$

In Chapter 2, we will discuss generalizations of Gronwall's inequality which extend to Volterra-type inequalities. First we consider some examples of the applications of Gronwall's inequality to differential equations.

1.3 Applications of Gronwall's Inequality

Gronwall's inequality is particularly useful when looking at the following first-order differential equation in $\mathbb{R}^n$: $y' = f(t, y)$ with f continuous and satisfying the Lipschitz condition

$$\|f(t, y) - f(t, z)\| \leq K\|y - z\|$$

for some $K \geq 0$. Gronwall's inequality gives the following error bound for approximate solutions of the initial-value problem [11]: If $z' = f(t, z)$, $\|y' - f(t, y)\| \leq \epsilon$, and $\|y(0) - z(0)\| \leq \delta$, then

$$\|y(t) - z(t)\| \leq \delta e^{Kt} + \epsilon \cdot \frac{e^{Kt} - 1}{K}, \quad t \geq 0,$$

where $\| \ \|$ is the norm on $\mathbb{R}^n$. This inequality is essential for observing that solutions depend continuously on the initial conditions, a key result in the theory of ordinary differential equations.

Another useful application is the following lemma. Rather than integral equations, it involves differential equations, but it has the same spirit as Gronwall's inequality. It will be used later in this thesis, so the proof is provided.

Lemma 1.3.1. *Let $y : [0, b] \to \mathbb{R}$ be a solution to $y' = f(t, y)$, where f is continuous and locally Lipschitz with respect to its second variable. Suppose that $w : [0, b] \to \mathbb{R}$ satisfies $w' \leq f(t, w)$ in $[0, b]$.*

(i) If $w(0) \leq y(0)$, then $w \leq y$ throughout $[0, b]$.

(ii) If $w(t) = y(t)$ for some $t \in (0, b]$, then $w = y$ throughout $[0, t]$.

Proof. (i) If we integrate $y'(t) = f(t, y)$, we have

$$y(t) - y(0) = \int_0^t f(s, y)ds,$$

or

$$y(t) = y(0) + \int_0^t f(s, y)ds.$$

Similarly, we have

$$w(t) \leq w(0) + \int_0^t f(s, w)ds.$$

Since $w(0) \leq y(0)$, the hypotheses of Theorem 1.1.2 are met, and we have $w(t) \leq y(t)$, $t \in [0, b]$.

(ii) Suppose $w(t_1) = y(t_1)$ for some $t_1 \in (0, b]$. Then

$$\begin{aligned} y'(t_1) &= f(t_1, y(t_1)) \\ &= f(t_1, w(t_1)) \\ &\leq w'(t_1)\,. \end{aligned}$$

Integrating over $[0, t_1]$, $y(t) \leq w(t)$ for some $t \in (0, t_1)$ (by the monotonicity of the Riemann integral). By (i), we have $y(t) \geq w(t)$ for $t < t_1$, which implies that $y(t) = w(t)$ for $t \in [0, t_1]$. ■

We note that Lemma 1.3.1 holds if $\leq$ is replaced with $\geq$.

The following chapters will lead to the following result about second-order differential equations:

Theorem 6.1.1 Let

$$\beta(p) = p^{\frac{-p}{2p-1}}(p-1)^{\frac{-p+1}{2p-1}}(2p-1)B\left(\tfrac{3p-1}{2p},\tfrac{1}{2}\right)^{\frac{2p}{2p-1}},$$

where

$$B(x,y) = \int_0^1 t^{x-1}(1-t)^{y-1}\,dt,$$

the beta function. Let u_q be the solution to the initial value problem

$$u'' + qu = 0\,, \qquad (u,u')(0) = (0,1) \tag{1.1}$$

where $q \in \mathcal{C}[0,b]$ such that the L^p-norm $\|q\|_p \le 1$. Then

(a) $\beta(p) = \inf\{z \in (0,b] : u_q(z) = 0 \text{ for some } q\}$

(b) Given $p > 1$ and $b \ge \beta(p)$, there is a unique function $q \in \mathcal{C}[0,b]$ such that the solution u_q to (1.1) vanishes at $\beta(p)$. Moreover, $q > 0$ on $(0,\beta(p))$ and $q = 0$ almost everywhere on the complement $[0,b] \setminus (0,\beta(p))$.

The following corollary gives a bound on the number of zeros to (1.1).

Corollary: On $[0,b]$, suppose we have the equation

$$u'' + qu = 0, \quad (u,u')(0) = (0,1),$$

with $q \in C[0,b]$, $q \ge 0$, and $||q||_p = m$. Then the number of zeroes n of a solution in $[0,b]$ is bounded:

$$n \le \left(\frac{b}{\beta(p)}\right)^{\frac{2p-1}{2p}} m^{1/2}.$$

The usefulness of Gronwall's inequality is widely recognized, and many mathematicians have set out to generalize it. The next chapter of this thesis will survey a few basic properties of integral operators as well as some Gronwall-type integral inequalities, and present a result of Stowe that is the inspiration for this thesis. The ultimate objective of this thesis is to present two new generalizations of Gronwall's inequality and exhibit their applications to differential inequalities and second-order differential operators.

CHAPTER 2
BASIC PROPERTIES OF INTEGRAL OPERATORS AND STOWE'S THEOREM

2.1 Basic Properties of Integral Operators

The following are some basic results about integral operators that will be assumed throughout this thesis.

Fix $b > 0$. Define $\Phi_{_K} : \mathcal{C}[0,b] \longrightarrow \mathcal{C}[0,b]$ by

$$(\Phi_{_K} u)(t) = \int_0^b K(t,s)u(s)\,ds,$$

where $K : [0,b] \times [0,b] \to \mathbb{R}$ is a function that agrees with continuous functions on each of the sets

$$\Delta = \{(t,s) : s \leq t\}\,, \quad \Delta' = \{(t,s) : s \geq t\}\,,$$

except, perhaps, along the diagonal $s = t$. The function $s \mapsto K(t,s)u(s)$ is then Riemann integrable whenever $t \in [0,b]$ and $u \in C[0,b]$. Let us allow K to be undefined on the diagonal and consider two such kernels J and K equal if they agree off the diagonal. Equivalently, we consider certain continuous functions on

$$\{(t,s) \in [0,b] \times [0,b] : s \neq t\}\,.$$

The following facts are straightforward to establish and thus presented without proof. The proofs can be found in *Basic Operator Theory* [9].

Theorem 2.1.1. *The function* $\Phi_K u : [0, b] \to \mathbb{R}$ *is continuous when* $u \in C[0, b]$.

Theorem 2.1.2. *The operator* Φ_K *is a bounded linear operator on* $C[0, b]$ *with respect to the norm* $\|u\|_\infty = \max |u|$, *and its operator norm satisfies* $\|\Phi_K\| \leq b\|K\|_\infty$, *where* $\|K\|_\infty = \operatorname{ess\,sup} |K|$.

Theorem 2.1.3. *The operator* Φ_K *is a positive operator if and only if* $I + \Phi_K$ *is a positive operator if and only if* $K \geq 0$.

Theorem 2.1.4. *If* Φ *is any bounded linear operator on* $C[0, b]$ *and*

$$\sum_{n=1}^{\infty} \|\Phi^n\| < \infty,$$

then $I - \Phi$ *is invertible. The sum converges if and only if* $\|\Phi^n\| < 1$ *for some* n.

Theorem 2.1.5. *The mapping* $K \mapsto \Phi_K$ *is linear and injective.*

Theorem 2.1.6. *The adjoint of* Φ_K *with respect to the inner product*

$$\langle u, v \rangle = \int_0^b uv$$

on $C[0, b]$ *is* Φ_{K^*},*where*$K^*(t, s) = K(s, t)$.

Theorem 2.1.7. *If* J *and* K *are kernels [of the kind that we are considering], then*

$$(J * K)(t, s) = \int_0^b J(t, r)K(r, s)\, dr, \quad (t, s) \in [0, b] \times [0, b],$$

is also a kernel [of the kind that we are considering], and $\Phi_{J*K} = \Phi_J \Phi_K$. *Furthermore, the product* $*$ *is associative but not commutative.*

Theorem 2.1.8. *If* J *and* K *vanish throughout* Δ', *then* $J * K$ *also vanishes throughout* Δ'.

Theorem 2.1.9. *If K vanishes throughout Δ', then the n-fold product $K * \cdots * K$ satisfies*

$$|(K * \cdots * K)(t,s)| \leq \|K\|_\infty^n \cdot \frac{(t-s)^{n-1}}{(n-1)!}, \quad (t,s) \in \Delta, \quad n \geq 0,$$

and $I - \Phi_K$ is invertible.

2.2 Motivation

It is natural to think about extending Gronwall's inequality to the setting of integral operators. In fact, this setting has been explored by many mathematicians because of the many applications to differential and integral equations. Below, are just a couple of examples of results containing inequalities that echo Gronwall's. For instance, H. Movljankulov and A. Filatov proved the following Gronwall-type inequality in this setting [4].

Theorem 2.2.1. *Let $u(t)$ be real, continuous, and nonnegative in $[0,b]$. Let*

$$u(t) \leq a(t) + b(t) \int_0^t k(t,s)u(s)\,ds,$$

where $a(t) \geq 0$, $b(t) \geq 0$, and $k(t,s) \geq 0$, are continuous functions for $0 \leq s \leq t \leq b$. Then

$$u(t) \leq A(t) \left[\exp B(t) \int_0^t K(t,s)\right] ds,$$

where

$$A(t) = \sup_{0 \leq s \leq t} a(s),$$

$$B(t) = \sup_{0 \leq s \leq t} b(s),$$

and

$$K(t,s) = \sup_{s \leq \sigma \leq t} k(\sigma, s).$$

Chu and Metcalf [7] proved the following generalization of Gronwall's inequality in this setting.

Theorem 2.2.2. *Let the functions u and f be continuous on the interval $[0, b]$ and let the function K be continuous and nonnegative on the triangle $0 \leq s \leq t \leq b$. If*

$$u(t) \leq f(t) + \int_0^t K(t,s)u(s)\,ds,$$

then

$$u(t) \leq f(t) + \int_0^t H(t,s)f(s)\,ds,$$

where $H(t,s) = \sum_{i=1}^{\infty} K_i(t,s), 0 \leq s \leq t \leq b$, is the resolvent kernel, and the K_i $(i = 1, 2, \ldots)$ are the iterated kernels of K.

These are just two examples of the type of results this thesis aims to achieve.

2.3 Overall Idea

The following discussion will present the overall idea for this thesis and give some results of Stowe [11].

Gronwall's inequality suggests the general question: For which $K \in C(\Delta)$ and which $B \in C[0,b]$ is the following implication valid?

$$\forall u, \phi \in C[0,b], \quad \left.\begin{array}{l} u \leq B + \Phi_K(u) \\ \phi = B + \Phi_K(\phi) \end{array}\right\} \Rightarrow u \leq \phi.$$

With $v = u - \phi$, the implication becomes

$$\forall w \in C[0,b], (I - \Phi_K)(w) \geq 0 \Rightarrow w \geq 0. \quad (*)$$

Theorem 2.3.1. *If $K \in C(\Delta)$, then $(I - \Phi_K)^{-1} = I + \Phi_R$ for a unique $R \in C(\Delta)$. The implication $(*)$ holds if and only if $R \geq 0$, and that occurs when $K \geq 0$. Furthermore, $R = K + (K * K) + (K * K * K) + \cdots$.*

Proof. If $J, K \in C(\Delta)$, then $\Phi_J \Phi_K = \Phi_{J*K}$, where

$$(J * K)(t,s) = \int_s^t J(t,r)K(r,s)\,dr, \qquad (t,s) \in \Delta.$$

The n-fold product satisfies

$$|(K * \cdots * K)(t,s)| \leq \|K\|_\infty^n (t-s)^{n-1}/(n-1)!.$$

If $J \in C(\Delta)$ and $u \in C[0,b]$, then

$$\|\Phi_J(u)\|_\infty \leq b\|J\|_\infty\|u\|_\infty.$$

From these, one concludes that the series

$$R = K + (K * K) + (K * K * K) + \cdots$$

converges uniformly in Δ and that $I + \Phi_R$ is a two-sided inverse of $I - \Phi_K$. Uniqueness follows from the fact that the mapping $J \mapsto \Phi_J$ is injective.

It is clear that $R \geq 0$ when $K \geq 0$ and that $I + \Phi_R$ is a positive operator when $R \geq 0$. Finally, suppose that $R < 0$ at some point (t, s). By continuity, one can choose that point to be in the interior of Δ and choose $\delta > 0$ such that

$$(s-\delta, s+\delta) \subseteq [0,t] \qquad \text{and} \qquad R(t,r) < 0 \qquad \text{for all} \qquad r \in (s-\delta, s+\delta).$$

If $v \in C[0,b]$ is positive in the interval $(s-\delta, s+\delta)$ and zero outside it, then the function $w = (I + \Phi_R)(v)$ satisfies $(I - \Phi_K)(w) = v \geq 0$, but

$$w(t) = 0 + \int_0^t R(t,r)v(r)\,dr < 0.$$

■

For the rest of this thesis, $R(K) := K + (K * K) + (K * K * K) + \cdots$. There is a slightly weaker requirement for $R \geq 0$ than $K \geq 0$. In the following proof I will use the notation $K^n = \underbrace{K * K * \cdots * K}_{\text{n times}}$.

Lemma 2.3.1. *If $K * K$ and $K + K * K$ are positive operators, then $R(K)$ is a positive operator.*

Proof. The proof is a straightforward algebraic manipulation of $R(K)$:

$$\begin{aligned} R(K) &= \lim_{n\to\infty} \sum_{k=1}^{n} K^k \\ &= \sum_{k=1}^{\infty} K^{2k-1} + K^{2k} \\ &= \sum_{k=1}^{\infty} K^{2k-2} * (K + K^2). \end{aligned}$$

Therefore if K^2 and $K + K^2$ are positive operators, $R(K)$ is a positive operator.

■

Suppose $u \in C^n[0, b]$ satisfies

$$L(u) = g(t), \quad E_0(u) = (\alpha_0, \dots, \alpha_{n-1}), \tag{2.1}$$

where g is continuous,

$$L(u) = u^{(n)} + \sum_{k=0}^{n-1} p_k(t) u^{(k)},$$

with p_k of class C^k, and

$$E_0(u) = \left(u, u', \dots, u^{(n-1)}\right)(0) \in \mathbb{R}^n.$$

Using Taylor's formula and integrating by parts, one finds that the solution of (2.1) satisfies the integral equation

$$u = B + \Phi_K(u), \quad (\Phi_K u)(t) = \int_0^t K(t,s)u(s)\,ds, \tag{2.2}$$

where

$$\begin{aligned} K(t,s) &= K_L(t,s) &&= \sum_{k=0}^{n-1}(-1)^{k+1}\frac{\partial^k}{\partial s^k}\left(\frac{(t-s)^{n-1}}{(n-1)!}p_k(s)\right), \\ B(t) &= B_{L,g,\alpha}(t) &&= \int_0^t \frac{(t-s)^{n-1}}{(n-1)!}g(s)\,ds + \sum_{k=0}^{n-1}\frac{\alpha_k}{k!}t^k \\ & &&+ \sum_{k=1}^{n-1}\sum_{j=0}^{k-1}(-1)^j\alpha_{k-1-j}\cdot\frac{\partial^j}{\partial s^j}\left(\frac{(t-s)^{n-1}}{(n-1)!}p_k(s)\right)\Bigg|_{s=0}. \end{aligned}$$

Since (2.2) has just one solution, it is equivalent to the initial-value problem. We can say more about K_L and $B_{L,g,\alpha}$:

Theorem 2.3.2 (Stowe's Theorem)**.** *If*

$$L(u) = u^{(n)} + \sum_{k=0}^{n-1} p_k(t)u^{(k)},$$

with p_k of class C^k, then the function $K = K_L$ is of the form

$$K(t,s) = \sum_{k=0}^{n-1}\frac{(t-s)^k}{k!}q_k(s),$$

q_k of class C^{n-1-k}, and when g is continuous and $\alpha \in \mathbb{R}^n$, the function $B = B_{L,g,\alpha}$ is of class C^n. These properties imply that the solution of $u = B + \Phi_K v$ solves

$$w^{(n)} + \sum_{k=0}^{n-1} p_k(t)v^{(k)} = g,$$

with $E_0(w) = \alpha$.

We call K_L the **Picard kernel** associated with L, since defining an iterative sequence by starting with a C^n-function u_0 satisfying $E_0(u_0) = \alpha$, inductively defining u_{m+1} as

$$u_{m+1} = B + \Phi_k u_m,$$

and letting $u = \lim u_m$, will converge to a solution.

In this setting, the operator L and initial conditions E_0 are associated with a unique Green's function, $G(t, s) \in C(\Delta)$. For such a $G(t, s)$, $\Phi_G(g)$ solves $L(u) = g$ and $E_0(u) = 0$ for $g \in C[0, b]$. Recall (2.2), which says a solution u satisfies

$$u = B + \Phi_K(u),$$

and thus

$$u - \Phi_K = B,$$

or

$$(I - \Phi_K)(u) = B.$$

If we apply $I+\Phi_{R(K)}$ to both sides of this equation the result is $u = \left(I + \Phi_{R(K)}\right) B$. Thus,

$$u(t) = \int_0^t \left[\frac{(t-s)^{n-1}}{(n-1)!} + \int_s^t \frac{(t-s)^{n-1}}{(n-1)!} [R(K)](t,r)\, dr \right] g(s)\, ds. \tag{2.3}$$

Since Green's functions are unique, the kernel in (2.3) must be the Green's function associated with the initial value problem, and we have the following theorem by Stowe [11].

Theorem 2.3.3. *The Green's function associated with the initial value problem*

$$L(u) = g(t), \quad E_0(u) = (\alpha_0, \dots, \alpha_{n-1}),$$

where g is continuous,

$$L(u) = u^{(n)} + \sum_{k=0}^{n-1} p_k(t) u^{(k)},$$

with p_k of class C^k, and

$$E_0(u) = (u, u', \dots, u^{(n-1)})(0) \in \mathbb{R}^n$$

is given by

$$G(t,s) = \frac{(t-s)^{n-1}}{(n-1)!} + \int_0^t R(k) \left(\frac{(t-s)^{n-1}}{(n-1)!} \right) ds.$$

In the next chapter, we will generalize the Gronwall's inequality strictly in terms of integral operators, and in Chapter 4 the focus will shift to applications to differential equations in which we will define the Taylor kernel, which is very similar to Stowe's Picard kernel, and discuss how the two are related.

CHAPTER 3
MAIN RESULTS

3.1 General Integral Theorem

To set up the general theorem, consider the following situation: Suppose that a continuous function ϕ on $[0, b]$ satisfies an equation

$$\phi(t) = C(t) + \int_0^t M(t,s)\phi(s)\,ds, \quad t \in [0,b],$$

where $C \in C[0, b]$ and M is a continuous function on the set

$$\Delta = \{(t,s) \in [0,b] \times [0,b] : s \le t\}.$$

In this thesis, the shorthand notation $\phi = C + \Phi_M(\phi)$ will be used frequently. If $u \in C[0, b]$ satisfies a similar inequality $u \ge B + \Phi_K(u)$, in certain situations one can deduce a relation of the form

$$(aI + \Phi_H)(u) \ge D + (aI + \Phi_H)(\phi)$$

for certain functions $H \in C(\Delta)$, $D \in C[0, b]$ and scalar a. That is,

$$au(t) + \int_0^t H(t,s)u(s)\,ds \ge D(t) + a\phi(t) + \int_0^t H(t,s)\phi(s)\,ds, \quad t \in [0,b].$$

In this chapter, we discuss conditions that are necessary and sufficient for such an implication to be valid.

In many applications, some or all of a, B, C, D, and H are zero. We include them in the main theorems in the interest of full generality. Notice that "$\geq$" has been used in both of the above inequalities, but the conditions also apply to implications involving "$\leq$" in both places since, for u, w, ϕ, $\psi \in C[0,b]$, the implications

$$\begin{aligned} &\phi = C + \Phi_M(\phi), \qquad u \geq B + \Phi_K(u) \qquad \Rightarrow \quad (aI + \Phi_H)(u) \geq D + (aI + \phi_H)(\phi) \\ &\psi = -C + \Phi_M(\psi), \quad w \leq -B + \Phi_K(w) \quad \Rightarrow \quad (aI + \Phi_H)(w) \leq -D + (aI + \psi_H)(\psi) \end{aligned}$$

are either both valid or both invalid. Again, the implication of interest is:

$$u \geq B + \Phi_K(u) \Rightarrow (aI + \Phi_H)(u) \geq D + (aI + \Phi_H)(\phi). \tag{3.1}$$

It is beneficial to manipulate this into an equivalent implication to provide a more straightforward proof for the first main theorem. The antecedent of (3.1) is equivalent to $(I - \Phi_K)(u) \geq B$. Adding $-(I - \Phi_K)(\phi)$ to both sides of the inequality gives

$$(I - \Phi_K)(u - \phi) \geq B + (\Phi_K - I)(\phi). \tag{3.2}$$

The consequent of (3.1) is equivalent to

$$(aI + \Phi_H)(u - \phi) \geq D.$$

Recall that

$$\{\left(I + \Phi_{R(K)}\right)(I - \Phi_K) = I,$$

so we can write the consequent as

$$(aI + \Phi_H)\left(I + \Phi_{R(K)}\right)(I - \Phi_K)(u - \phi) \geq D. \tag{3.3}$$

Let $w = (I - \Phi_K)(u - \phi)$. Substitute w into (3.2) and (3.3) to get

$$w \geq B + (\Phi_K - I)(\phi)$$

and

$$(aI + \Phi_H)\left(I + \Phi_{R(K)}\right)(w) \geq D,$$

respectively. Thus, the main question is equivalent to: What conditions on a, ϕ, K, M, H, B, C, and D are necessary and sufficient for the implication

$$w \in C[0, b] \quad \text{and} \quad w \geq B + (\Phi_K - I)(\phi) \Rightarrow (aI + \Phi_H)\left(I + \Phi_{R(K)}\right)(w) \geq D \tag{3.4}$$

to hold? This preliminary result will be used in the proof of the following theorem, which provides necessary and sufficient conditions on a, ϕ, K, M, H, B, C, and D for (3.4) to hold.

Theorem 3.1.1. *Let a be a real number and let functions B, C, D, $\phi \in C[0, b]$. Let H, M, and $K \in C[\Delta]$. Suppose $\phi = C + \Phi_M(\phi)$. The implication*

$$u \in C[0, b] \quad \textit{and} \quad u \geq B + \Phi_K(u) \Rightarrow (aI + \Phi_H)(u) \geq D + (aI + \Phi_H)(\phi) \tag{3.5}$$

holds if and only if the following hold:

(i) $(aI + \Phi_H)\left[\left(I + \Phi_{R(K)}\right)(B - C + \Phi_{K-M}(\phi))\right] \geq D$, *and*

(ii) $(aI + \Phi_H)\left(I + \Phi_{R(K)}\right)$ *is a positive operator.*

Proof. Suppose conditions (i) and (ii) hold. Let $u \in C[0,b]$ with $u \geq B + \Phi_K(u)$. As in the discussion above, let $w = (I - \Phi_K)(u - \phi)$, and consider the equivalent implication (3.4). Then $w \geq B + (\Phi_K - I)(\phi)$ and

$$\begin{aligned}(aI + \Phi_H)\left(I + \Phi_{R(K)}\right)(w) &\geq (aI + \Phi_H)\left(I + \Phi_{R(K)}\right)[B + (\Phi_K - I)(\phi)]\\ &= (aI + \Phi_H)\left(I + \Phi_{R(K)}\right)[B - C + \Phi_{K-M}(\phi)]\\ &\geq D.\end{aligned}$$

Now suppose condition (i) fails. Then $(aI + \Phi_H)\left(I + \Phi_{R(K)}\right)[(B - C + \Phi_{K-M}(\phi)](t) < D(t)$ for some $t \in [0,b]$.

Let $w = B + (\Phi_K - I)\phi$. Then the hypotheses of the theorem are met and

$$\begin{aligned}(aI + \Phi_H)\left(I + \Phi_{R(K)}\right)(w)(t) &\leq (aI + \Phi_H)\left(I + \Phi_{R(K)}\right)[B + (\Phi_K - I)\phi](t)\\ &= (aI + \Phi_H)\left(I + \Phi_{R(K)}\right)[(B - C + \Phi_{K-M}(\phi)](t)\\ &< D(t).\end{aligned}$$

Suppose (ii) fails. Then there exists $v \in C[0,b]$ such that $v \geq 0$ and

$$(aI + \Phi_H)\left(I + \Phi_{R(K)}\right)(v)(t) < 0$$

for some $t \in [0,b]$. Let

$$w = B + (\Phi_K - I)\phi + kv,$$

where k is sufficiently large to insure that

$$(aI + \Phi_K)\left(I + \Phi_{R(K)}\right)(w)(t) + (aI + \Phi_H)\left(I + \Phi_{R(K)}\right)(kv)(t) < D(t).$$

Then the hypotheses of the theorem are met and

$$\begin{aligned}(aI + \Phi_H)\left(I + \Phi_{R(K)}\right)(w)(t) &= (aI + \Phi_H)\left(I + \Phi_{R(K)}\right)[B + (\Phi_K - I)\phi + kv](t) \\ &= (aI + \Phi_K)\left(I + \Phi_{R(K)}\right)[B - C + \Phi_{K-M}(\phi) + kv](t) \\ &< D(t),\end{aligned}$$

which implies that (3.4) is false. ■

Remark 3.1.1. *Note that Stowe's Theorem is a special case of Theorem 3.1.1 with* $B = C$*,* $M = K$*,* $H = 0$*, and* $a = 1$*.*

3.2 General Integral Comparison Theorem

Theorem 3.1.1 gives us necessary and sufficient conditions for (3.5) to hold. While this is useful, it is not always practical to apply. The following theorem supplies sufficient conditions for (3.5), which leads to many useful results.

Theorem 3.2.1. *Let functions* B*,* C*,* D*,* u*,* $\phi \in C[0, b]$*. Let* a *be a real number, and let* H*,* M*,* $K \in C(\Delta)$*. Suppose*

$$\phi(t) = C(t) + \Phi_M(\phi)(t) \quad \textit{and} \quad u(t) \geq B(t) + \Phi_K(u)(t).$$

Define

$$\Phi := (aI = \Phi_H) \circ \left(I + \Phi_{R(M)}\right).$$

Suppose

$$\Phi \circ \Phi_{[K-M]} = \Phi_\Theta \circ (aI + \Phi_H)$$

for some $\Theta \in C(\Delta)$*. If* Φ *and* $I + \Phi_{R(\Theta)}$ *are positive, and*

$$\left(I + \Phi_{R(\Theta)}\right)[\Phi(B - C + \Phi_{K-M}(\phi))] \geq D,$$

then

$$(aI + \Phi_H)(u) \geq D + (aI + \Phi_H)(\phi).$$

Proof. Since $\Phi = (aI + \Phi_H) \circ \left(I + \Phi_{R(M)}\right)$, we have

$$(aI + \Phi_H)(u - \phi) = \Phi[(I - \Phi_M)(u - \phi)] = \Phi[u - \Phi_M(u) - \phi + \Phi_M(\phi)].$$

Now, since Φ is positive,

$$\phi(t) = C(t) + \Phi_M(\phi)(t)$$

and

$$u(t) \geq B(t) + \Phi_K(u)(t)$$

imply that

$$(aI + \Phi_H)(u - \phi) \geq \Phi[B + \Phi_K(u) - \Phi_M(u) - C].$$

Adding and subtracting $\Phi_K(\phi)$ from the right side, it follows that

$$\begin{aligned}(aI + \Phi_H)(u - \phi) &\geq \Phi[B - C + \Phi_{K-M}(u - \phi) + \Phi_{K-M}(\phi)] \\ &= \Phi[B - C + \Phi_{K-M}(u - \phi)] + \Phi[\Phi_{K-M}(\phi)] \\ &= \Phi[B - C + \Phi_{K-M}(u - \phi)] + \Phi_\Theta[(aI + \Phi_H)(\phi)].\end{aligned}$$

Subtracting $\Phi_\Theta[(aI + \Phi_H)(\phi)]$ from both sides implies that

$$(I - \Phi_\Theta)[(aI + \Phi_H)(u - \phi)] \geq \Phi_G[B - C + \Phi_{K-M}(\phi)]$$

and

$$(aI + \Phi_H)(u - \phi) \geq (I + \Phi_{R_\Theta})[\Phi_G[B - C + \Phi_{K-M}(\phi)]] \geq D.$$

Therefore $(aI + \Phi_H)(u) \geq D + (aI + \Phi_H)(\phi)$. $\blacksquare$

CHAPTER 4

APPLICATIONS TO LINEAR DIFFERENTIAL EQUATIONS

In this chapter, we will discuss applications of Theorems 3.1.1 and 3.2.1 to linear differential equations with initial conditions. The first few results establish a basic equivalence between such differential inequalities and specific integral inequalities of the form investigated in Chapter 3. This will allow the translation of Theorems 3.1.1 and 3.2.1 into theorems about differential inequalities. With these translations, we will see useful applications to n^{th} order differential equations as well as a generalization of Sturm's Comparison Theorem in the 2^{nd} order case. The discussion will start with few simple but useful results.

Note: Throughout this chapter, let

$$H_n(t,s) = \frac{(t-s)^{n-1}}{(n-1)!},$$

the Green's function associated with the differential operator $L(v) = v^{(n)}$.

Lemma 4.0.1. *Suppose $v \in C^n[0,b]$ with initial values $v^{(k)}(0) = \alpha_k$, $0 \le k \le n$. Then*

$$v(t) = \sum_{k=0}^{n} \alpha_k \frac{t^k}{k!} + \int_0^t v^{(n)}(s) \frac{(t-s)^n}{(n)!} ds.$$

Proof. We proceed by induction. Let $n = 1$. Then integrating v' results in

$$\int_0^t v'(s)ds = v(t) - \alpha_0$$

and

$$v(t) = \int_0^t v'(s)ds + \alpha_{n-1}.$$

Suppose the statement holds for some n and let $v \in C^{n+1}[0,b]$. Then

$$v(t) = \sum_{k=0}^{n} \alpha_k \frac{t^k}{k!} + \int_0^t v^{(n)}(s) \frac{(t-s)^n}{(n)!} ds,$$

and integrating by parts gives

$$\begin{aligned} v(t) &= \sum_{k=0}^{n} \alpha_k \frac{t^k}{k!} + \int_0^t v^{(n+1)}(s) \frac{(t-s)^{n+1}}{(n+1)!} ds + \alpha_{n+1} \frac{t^{n+1}}{(n+1)!} \\ &= \sum_{k=0}^{n+1} \alpha_k \frac{t^k}{k!} + \int_0^t v^{(n+1)}(s) \frac{(t-s)^{n+1}}{(n+1)!} ds. \end{aligned}$$

Therefore, by the Principle of Mathematical Induction, the statement holds for all $n \geq 1$. ■

Remark 4.0.1. *Applying Lemma 4.0.1 to $v^{(m)}$, where $m < n$, provides the following formula:*

$$v^{(m)}(t) = \sum_{k=m}^{n-1} \alpha_k \frac{t^{k-m}}{(k-m)!} + \int_0^t v^{(n)}(s) \frac{(t-s)^{n-m}}{(n-m)!} ds.$$

This will be the building block for translating differential inequalities to integral inequalities.

Theorem 4.0.1. *Let $v \in C^n[0,b]$, $g \in C[0,b]$, and let ℓ be the differential operator defined by*

$$\ell(v) = v^{(n)} + \sum_{k=0}^{n-1} p_k v^{(k)},$$

where the p_k are continuous for $0 \leq k < n$. Suppose further that the initial conditions $v^{(k)}(0) = \alpha_k$ hold for $0 \leq k < n$. Then $l(v) \geq g$ if and only if

$$v^{(n)} \geq B + \int_0^t K(t,s) v^{(n)}(s)\, ds,$$

where

$$B(t) = g(t) - \sum_{k=0}^{n-1} p_k \left[\sum_{j=k}^{n-1} \frac{\alpha_j t^{j-k}}{(j-k)!} \right] \quad \textit{and}$$

$$K(t,s) = -\sum_{k=0}^{n-1} p_k(t) \frac{(t-s)^{n-k-1}}{(n-k-1)!}.$$

Proof. First, $\ell(v) \geq g$ if and only if

$$v^{(n)}(t) \geq g(t) - \sum_{k=0}^{n-1} p_k v^{(k)}(t).$$

Applying Lemma 4.0.1 to the right hand side of the inequality gives

$$\begin{aligned}
v^{(n)}(t) &\geq g(t) - \sum_{k=0}^{n-1} p_k(t)\left[\left(\sum_{j=k}^{n-1}\alpha_j \frac{t^{j-k}}{(j-k)!}\right) + \int_0^t v^{(n)}(s)\frac{(t-s)^{n-k}}{(n-k)!}ds\right] \\
&= g(t) - \sum_{k=0}^{n-1} p_k(t)\left[\sum_{j=k}^{n-1}\alpha_j \frac{t^{j-k}}{(j-k)!}\right] - \sum_{k=0}^{n-1} p_k(t)\left[\int_0^t v^{(n)}(s)\frac{(t-s)^{n-k}}{(n-k)!}ds\right] \\
&= B(t) - \int_0^t \left[\sum_{k=0}^{n-1} p_k(t) v^{(n)}(s)\frac{(t-s)^{n-k}}{(n-k)!}ds\right] \\
&= B(t) + \int_0^t K(t,s)v^{(n)}(s)\,ds.
\end{aligned}$$

Therefore $\ell(v) \geq g$ if and only if

$$v^{(n)} \geq B + \int_0^t K(t,s)v^{(n)}(s)\,ds.$$

■

Notice, if we apply Theorem 4.0.1 to Lemma 1.3.1, the following lemma, which will be used in a later chapter, is immediate:

Lemma 4.0.2. *Let $y : [0,b] \to \mathbb{R}$ and $w : [0,b] \to \mathbb{R}$ satisfy*

$$y = \int_0^t f(s,y)\,ds$$

and

$$w \geq \int_0^t f(s,w)\,ds,$$

where $f(s,t)$ is continuous and locally Lipschitz with respect to t.

(i) If $w(0) \geq y(0)$, then $w(t) \geq y(t)$ throughout $[0,b]$.

(ii) If $w(t) = y(t)$ for some $t \in (0,b]$, then $w(t) = y(t)$ throughout $[0,t]$.

Definition 4.0.1. *We call K in Theorem 4.0.1 the* ***Taylor kernel*** *associated with the differential operator ℓ.*

The following discussion will set up the next theorem, a result that is used quite often. Let

$$L(v) = v^{(n)} + \sum_{k=0}^{n-1} p_k v^{(k)} \text{ on } [0,b],$$

where $p_k \in C[0,b]$, $k = 0,1,\ldots,n-1$, and $n \geq 2$. Consider the solution v to $L(v) = \delta_a$, where δ_a is the Dirac-delta function for $a \in (0,b)$. It is generally known that v solves $L(v) = 0$ in $(0,a]$ and $(a,b]$ separately, is of class C^n in each of those intervals, and is C^{n-2} overall. We will prove this using Theorem 3.2.1.

In order to translate Theorems 3.1.1 and 3.2.1 into theorems involving differential operators, an equivalence between implications of the form

$$L(\psi) = h \quad \text{and} \quad \ell(v) \geq g \implies v \geq f + \psi$$

and an integral implication of the form investigated in Chapter 3,

$$\phi = C + \Phi_M(\phi) \quad \text{and} \quad u \geq B + \Phi_K(u) \implies (aI + \Phi_H)(u) \geq D + (aI + \Phi_H)(\phi),$$

is needed. Theorem 4.0.1 and Lemma 4.0.1 provide the desired equivalence.

Theorem 4.0.2. *Let ℓ, v, g, K, and B be as in Theorem 4.0.1. Let $\psi \in C^n[0,b]$, f, $h \in C[0,b]$, and L be the differential operator defined by*

$$L(\psi) = \psi^{(n)} + \sum_{k=0}^{n-1} P_k \psi^{(k)},$$

where P_k is continuous for $0 \le k < n$. Suppose further that initial conditions $\psi^{(k)}(0) = \beta_k$ for $0 \le k < n$ hold. The following implications are equivalent:

- *If $L(\psi) = h$ and $\ell(v) \ge g$, then $v \ge f + \psi$.*
- *If $\psi^{(n)} \ge C + \Phi_M\left(\psi^{(n)}\right)$ and $v^{(n)} \ge B + \Phi_K\left(v^{(n)}\right)$, then $\Phi_{H_n}\left(v^{(n)}\right) \ge D + \Phi_{H_n}\left(\psi^{(n)}\right)$,*

where

$$C(t) = h(t) - \sum_{k=0}^{n-1} P_k \left[\sum_{j=k}^{n-1} \frac{\beta_j t^{j-k}}{(j-k)!}\right],$$

$$D(t) = f(t) + \sum_{k=0}^{n-1} (\beta_k - \alpha_k) \frac{t^k}{k!},$$

and $M(t,s)$ is the Taylor kernel associated with L.

Proof. By Theorem 4.0.1, $L(\psi) = h$ and $\ell(v) \ge g$ are equivalent to

$$\psi^{(n)} = C + \Phi_M\left(\psi^{(n)}\right)$$

and

$$v^{(n)} \ge B + \Phi_K\left(v^{(n)}\right),$$

respectively. Lemma 4.0.1 implies $v \geq f + \psi$ is equivalent to

$$\Phi_{H_n}\left(v^{(n)}\right) \geq D + \Phi_{H_n}\left(\psi^{(n)}\right).$$

Therefore the two implications are interchangeable. ■

Theorem 4.0.2 states that any linear differential inequality can be associated with an equivalent integral inequality. This allows one to apply Theorems 3.1.1 and 3.2.1 to differential inequalities. In the next section, we will translate Theorem 3.1.1 into an equivalent statement about differential inequalities.

4.1 Translation of Theorem 3.1.1

The next theorem is the translation of Theorem 3.1.1 into a statement about linear differential operators.

Theorem 4.1.1. *Let L and ℓ be the linear differential operators defined by*

$$L(u) = u^{(n)} + \sum_{k=0}^{n-1} u^{(k)} P_k$$

and

$$\ell(u) = u^{(n)} + \sum_{k=0}^{n-1} u^{(k)} p_k,$$

with $p_k,\ P_k \in C[0,b]$ *for all* k*. Let* $f,\ g,\ h \in C[0,b]$*, and let* $\psi,\ v \in C^n[0,b]$ *with initial conditions* $\psi^{(k)}(0) = \beta_k$ *and* $v^{(k)}(0) = \alpha_k$ *for* $0 \leq k \leq n$*. The implication*

$$\textit{If } L(\psi) = h \textit{ and } \ell(v) \geq g, \textit{ then } v \geq f + \psi$$

holds if and only if

$$\Phi_{H_n}\left(I + \Phi_{R(K)}\right)\left(B - C + \Phi_{K-M}\left(\psi^{(n)}\right)\right) \geq D$$

and $\Phi_{H_n}\left(I + \Phi_{R(K)}\right)$ *is a positive operator where* $K(t,s)$ *and* $M(t,s)$ *are the Taylor kernels associated with* ℓ *and* L*, respectively, and*

$$B(t) = g(t) - \sum_{k=0}^{n-1} p_k \left[\sum_{j=k}^{n-1} \frac{\alpha_j t^{j-k}}{(j-k)!}\right],$$

$$C(t) = h(t) - \sum_{k=0}^{n-1} P_k \left[\sum_{j=k}^{n-1} \frac{\beta_j t^{j-k}}{(j-k)!}\right],$$

and

$$D(t) = f(t) + \sum_{k=0}^{n-1} (\beta_k - \alpha_k) \frac{t^k}{k!}.$$

Proof. Applying Theorem 4.0.2 to

$$L(\psi) = h \text{ and } \ell(v) \geq g \implies v \geq f + \psi$$

gives the equivalent statement

$$\psi^{(n)} = C+\Phi_M\left(\psi^{(n)}\right) \quad \text{and} \quad v^{(n)} \geq B+\Phi_K\left(v^{(n)}\right) \implies \Phi_{H_n}\left(v^{(n)}\right) \geq D+\Phi_{H_n}\left(\psi^{(n)}\right).$$

Now apply Theorem 3.1.1 to get the desired result. ■

4.2 Translation of Theorem 3.2.1

In this section, we will translate Theorem 3.2.1 into a theorem about linear differential operators. It is first necessary to discuss the Green's functions associated with linear differential operators with initial conditions. It is natural for Green's functions to come up in this context as the main discussion focuses on comparing solutions to different linear differential operators with initial conditions. The following proposition gives a relation between the Green's function and the Taylor kernel associated with an operator ℓ. The theorem parallels Theorem 2.3.3 which gives a similar relation for the Picard kernel.

Proposition 4.2.1. *Let ℓ be the linear operator given by*

$$\ell(u) = u^{(n)} + \sum_{k=0}^{n-1} u^{(k)} p_k,$$

with p_k continuous on $[0, b]$, and let K be the Taylor operator associated with ℓ. If $G(t, s)$ is the Green's function for ℓ on Δ, then $G = H_n + H_n * R(K)$.

Proof. If $G(t, s)$ is the Green's function for ℓ, then, for any continuous function g, $v = \Phi_G(g)$ satisfies $l(v) = g$ and

$$v^{(n)} = g(t) + \Phi_K\left(v^{(n)}\right).$$

We also have

$$v = \Phi_{H_n}\left(v^{(n)}\right).$$

Thus

$$v = \Phi_{H_n}\left(v^{(n)}\right) = \Phi_G(g) = \Phi_G\left(v^{(n)} - \Phi_K\left(v^{(n)}\right)\right)$$

and, in particular,

$$\Phi_{H_n}\left(v^{(n)}\right) = \Phi_G\left(v^{(n)} - \Phi_K\left(v^{(n)}\right)\right)$$

or

$$\Phi_{H_n}\left(v^{(n)}\right) = \Phi_G(I - \Phi_K)\left(I + v^{(n)}\right).$$

Applying $\left(I + \Phi_{R(K)}\right)$ on the right, results in $\Phi_{H_n}\left(I + \Phi_{R(K)}\right)\left(v^{(n)}\right) = \Phi_G\left(v^{(n)}\right)$. As this is true for any g (and, hence, any $v^{(n)}$),

$$G = H_n * (I + R(K)) = G = H_n + H_n * R(K).$$

$\blacksquare$

Because of the uniqueness of Green's functions, we can now relate the Taylor kernel to the Picard kernel in an interesting way.

Theorem 4.2.1. *Let ℓ be the linear operator given by*

$$\ell(u) = u^{(n)} + \sum_{k=0}^{n-1} u^{(k)} p_k,$$

with p_k continuous on $[0, b]$, and let K and K_p be the Taylor and Picard kernels associated with ℓ, respectively. Then

$$R(K_p) * H_n = H_n * R(K).$$

Proof. Since Green's functions are unique, we have

$$H_n * (I + R(K)) = (I + R(K_p)) * H_n$$

by 4.2.1 and 2.3.3. Therefore

$$R(K_p) * H_n = H_n * R(K).$$

∎

Suppose we are looking at the Taylor kernels K and M associated with the differential operators

$$\ell(v) = \sum_{k=0}^{n-1} p_k(t) v^{(k)}(t)$$

and

$$L(v) = \sum_{k=0}^{n-1} P_k(t) v^{(k)}(t),$$

respectively, with P_k, $p_k \in C^k[0,b]$ for $0 \le k \le n-1$. Let G be the Green's function for the operator L. The following lemma will give a continuous function $\Theta_n(t,s)$ so that

$$G * [K - M] = \Theta_n * H_n,$$

as in Theorem 3.2.1.

Lemma 4.2.1. *Let*

$$\Theta_n = \sum_{k=0}^{n-1} \frac{\partial^k}{\partial s^k}\Big[G(t,s)(P_k(s) - p_k(s))\Big].$$

Then

$$G * [K - M] = \Theta_n * H_n.$$

Proof. First,

$$\begin{aligned} G * [K - M] &= \int_s^t \sum_{k=0}^{n-1} G(t,r)\Big[P_k(r) - p_k(r)\Big]\frac{(r-s)^k}{k!}\,dr \\ &= \sum_{k=0}^{n-1} \int_s^t G(t,r)\Big[P_k(r) - p_k(r)\Big]\frac{(r-s)^k}{k!}\,dr. \end{aligned}$$

Letting

$$u = G(t,r)[P_k(r) - p_k(r)]$$

and

$$dv = (r-s)^k dr/k!$$

and integrating by parts k times, we get

$$\begin{aligned}\sum_{k=0}^{n-1}\int_s^t G(t,r)[P_k(r)-p_k(r)]\frac{(r-s)^k}{k!}\,dr &= \sum_{k=0}^{n-1}\int_s^t \frac{\partial^k}{\partial r^k}\Big[G(t,r)(P_k(r)-p_k(r))\Big](r-s)\,dr\\ &= \int_s^t \sum_{k=0}^{n-1}\frac{\partial^k}{\partial r^k}\Big[G(t,r)(P_k(r)-p_k(r))\Big](r-s)\,dr\\ &= \Theta_n * H_n.\end{aligned}$$

Therefore

$$G*[K-M] = \Theta_n * H_n.$$

■

Note: Henceforth, Θ_n will be defined as in Lemma 4.2.1.

Theorem 4.2.2. *Let ℓ and L be differential operators defined by*

$$\ell(u) = \sum_{k=0}^{n-1} p_k(t)u^{(k)}(t)$$

and

$$L(u)\sum_{k=0}^{n-1} P_k(t)u^{(k)}(t),$$

respectively, with P_k, $p_k \in C^k[0,b]$ *for* $0 \le k \le n-1$. *Suppose* $L(\psi) = h$ *and* $\ell(v) \ge g$ *for* h, $g \in C[0,b]$ *and* ψ, $v \in C^{(n)}[0,b]$, *with initial conditions* $v^{(k)}(0) = \alpha_k$ *and* $\psi^{(k)}(0) = \beta_k$ *for* $0 \le k \le n$. *Let* $f \in C[0,b]$, *let* $G(t,s)$ *be the Green's function associated with* L, *and let* $K(t,s)$ *and* $M(t,s)$ *be the Taylor kernels associated with* ℓ *and* L, *respectively. Let*

$$B(t) = g(t) - \sum_{k=0}^{n-1} p_k \left[\sum_{j=k}^{n-1} \frac{\alpha_j t^{j-k}}{(j-k)!} \right],$$

$$C(t) = h(t) - \sum_{k=0}^{n-1} P_k \left[\sum_{j=k}^{n-1} \frac{\beta_j t^{j-k}}{(j-k)!} \right],$$

and

$$D(t) = f(t) + \sum_{k=0}^{n-1} (\beta_k - \alpha_k) \frac{t^k}{k!}.$$

If Φ_G *and* $I + \Phi_{R(\Theta_n)}$ *are positive and*

$$(I + \Phi_{R(\Theta_n)}) \circ \Phi_G(B - C + \Phi_{K-M}\left(\psi^{(n)}\right)) \ge D,$$

then $v \ge f + \psi$.

Proof. By Lemma 4.0.1, $L(\psi) = h$ is equivalent to

$$\psi^{(n)}(t) = C(t) + \Phi_M\left(\psi^{(n)}\right)(t)$$

and $\ell(v) \geq g$ is equivalent to

$$v^{(n)}(t) \geq B(t) + \Phi_K\left(v^{(n)}\right)(t).$$

By Lemma 4.2.1,

$$G * [K - M] = \Theta * H_n.$$

Thus, all the hypotheses of Theorem 3.2.1 are met and

$$(aI + \Phi_{H_n})(u) \geq (aI + \Phi_{H_n})\left(\psi^{(n)}\right) + D.$$

Applying Lemma 4.0.1 again, $v \geq k + \psi$. ■

4.3 Picard Iteration

Here we will give an overview of Picard iteration and present two similar iteration techniques for n^{th}-order linear initial value problems using the Taylor and Picard kernels.

Picard iteration is a numerical method used to construct solutions to the differential equation $y' = f(x, y)$ with initial condition $y(0) = y_0$, where f satisfies a Lipschitz condition. The first step is to convert the differential equation into the corresponding integral equation

$$y(x) = y_0 + \int_0^x f[s, y(s)]\, ds.$$

The iterative process starts with $y_0(x) = y_0$ and continues with

$$y_n(x) = y_0 + \int_0^x f[s, y_{n-1}(s)]\, ds.$$

This sequence converges and $y(x) = \lim_{n\to\infty} u_n(x)$ is a solution to the initial value problem [8]. While Picard iteration is not often practically used, it has been used to prove some powerful and useful results such as the *Existence and Uniqueness* Theorem which is commonly used [8].

The Picard and Taylor kernels can be used in an iterative technique which mirrors the Picard method. Given an n^{th}-order linear initial value problem,

$$L(v) = \sum_{k=0}^{n-1} p_k(t) v^{(k)}(t) = g(t), \qquad v^{(k)}(0) = \alpha_k \quad \text{for} \quad 0 \le k \le n-1, \tag{4.1}$$

with p_k of class C^k and g continuous, the first step for the iterative technique is to convert the differential equation into the corresponding integral equation

$$v^{(n)} = B + \int_0^t K(t,s) v^{(n)}(s)\, ds.$$

Here, as in Theorem 4.0.1,

$$B(t) = g(t) - \sum_{k=0}^{n-1} p_k \left[\sum_{j=k}^{n-1} \frac{\alpha_j t^{j-k}}{(j-k)!} \right],$$

and K is the Taylor kernel associated with L. In fact, by Theorem 2.3.1, we could also replace the Taylor kernel with the Picard kernel and use the $B(t)$ given there. Keeping that in mind, the following discussion applies to both kernels. Let $v_0(t)$ be a C^n-function which satisfies the initial conditions. Since all the functions involved are relatively well-behaved, the iteration defined by

$$v_n(t) = B(t) + \int_0^t K(t,s)v_{n-1}(s)\,ds$$

for $n \geq 1$ will converge to the solution of (4.1).

CHAPTER 5

STURM'S COMPARISON THEOREM

5.1 Classical Sturm's Comparison Theorem and a Generalization

Using what has previously been established, it is natural to consider the location of zeros of solutions to the differential equations we examined in Chapter 4 in order to arrive at comparison theorems similar to the celebrated Sturm's Comparison Theorem. Sturm discovered his theorem in 1836, and at the time, his work on finding roots and oscillation were considered very unique [13]. Now, Sturm's ideas and techniques are indispensable in functional analysis.

Here, I will state the portion of the theorem that is appropriate for this discussion and give a new proof.

Theorem 5.1.1. *Suppose continuous functions $\psi(t)$ and $v(t)$ satisfy the following conditions:*

$$\psi'' + Q\psi = 0 \ \ \textit{and} \ \ v'' + qv = 0, \ t \in [0, b],$$

$$(\psi, \psi')(0) = (\alpha_1, \alpha_2),$$

$$(v, v'')(0) = (\beta 1, \beta_2),$$

and

$$\beta_1/\beta_2 \geq \alpha_1/\alpha_2,$$

where $Q, q \in C[0,b]$ *and* $Q \geq q$*. If* ψ *has no zero in* $(0,b)$*, then* v *has no zero in* $(0,b)$*.*

Proof. If ψ has no zero in $(0,b)$, we can assume that it is positive, for if it were negative, we could just take the opposite sign, and the following argument would still hold. Let

$$y(t) := \psi(t)\psi(s)\int_s^t 1/\psi^2(r)\,dr$$

for each $s \in (0,b)$ with $s < t$. Then

$$y'(t) = \psi(s)/\psi(t) + \psi(s)\psi'(t)\int_s^t 1/\psi^2(r)\,dr$$

and

$$y''(t) = \psi(s)\psi''(t)\int_s^t 1/\psi^2(r)\,dr.$$

Thus, $y(t)$ satisfies $y'' + Qy = 0$ since

$$
\begin{aligned}
y''(t) + Q(t)y(t) &= y''(t) - \frac{\psi''(t)}{\psi(t)} y(t) \\
&= \psi(s)\psi''(t) \int_s^t \frac{1}{\psi^2(r)}\, dr - \frac{\psi''(t)}{\psi(t)} \left[\psi(t)\psi(s) \int_s^t \frac{1}{\psi^2(r)}\, dr \right] \\
&= \psi(s)\psi''(t) \int_s^t \frac{1}{\psi^2(r)}\, dr - \psi(s)\psi''(t) \int_s^t \frac{1}{\psi^2(r)}\, dr \\
&= 0.
\end{aligned}
$$

We also have $(y, y')(s) = (0, 1)$ for each $s \in (0, b)$, so

$$
G(t, s) := \psi(t)\psi(s) \int_s^t 1/\psi^2(r)\, dr
$$

is a Green's function for the differential equation that ψ satisfies. Since ψ is positive, so is Φ_G.

Now, suppose we have a function ϕ that satisfies

$$
\phi'' + Q\phi = 0
$$

and

$$
(\phi, \phi')(0) = (\beta_0, \beta_1)
$$

on $[0, b]$. Assume both α_0 and β_0 are positive and let $c = \beta_0/\alpha_0$. Consider the function $c\psi$. Note that

$$
(c\psi)(0) = \alpha_0(\beta_0/\alpha_0) = \beta_0 = \phi(0).
$$

We also have

$$(c\psi)'(0) = \alpha_1(\beta_0/\alpha_0) \leq \beta_1$$

by the statement of the theorem. Thus,

$$(c\psi)'(0) \leq \phi'(0)$$

and

$$\phi - c\psi = dG(t, 0)$$

on $[0, b]$ for some $d \geq 0$. Therefore, ϕ has no zero in $(0, b)$.

Finally, let us consider the function v. Note that

$$v'' + Qv = (Q - q)v.$$

Thus,

$$v = \phi + \Phi_G[(Q - q)v]$$

and, since everything involved is positive, $v \geq \phi$ on $[0, b]$, implying that v has no zero in $(0, b)$.

■

The next theorem is a generalized comparison theorem similar to Sturm's Theorem. First, we need a lemma.

Lemma 5.1.1. *Suppose a continuous function u satisfies*

$$l(v) = v^{(n)} + \sum_{k=0}^{n-1} p_t v^{(k)} = 0$$

on $[0, b]$, with p_k continuous on $[0, b]$ and initial conditions $v^{(k)} = 0$ for $0 \leq k \leq n-1$ and $v^{(n)} = 1$. Suppose further, a continuous function ψ satisfies

$$L(\psi) = \psi^{(n)} + \sum_{k=0}^{n-1} p_t \psi^{(k)} = 0$$

on $[0, b]$, with P_k continuous on $[0, b]$ and initial conditions $\psi^{(k)} = 0$ for $0 \leq k \leq n-1$ and $\psi^{(n)} = 1$. Let G be the Green's function associated with L. Then

$$\Phi_G\left[\sum_{k=0}^{n-1}(P_k - p_k)\frac{t^{n-k-1}}{(n-1-k)!}\right] = \Phi_{\theta_n}\left[\frac{t^{n-1}}{(t-1)!}\right].$$

Proof. Proof by induction. Let $n = 1$. G in this case is the constant function 1 and

$$\Phi_G[P_0 - p_0] = \int_0^t P_0(s) - p_0(s)\, ds = \Phi_{\theta_1}(1).$$

Suppose the statement holds for some n. Then, looking at the $n+1$ case,

$$\begin{aligned}\Phi_G\left[\sum_{k=0}^{n-1}(P_k - p_k)\frac{t^{n-k}}{(n-1)!}\right] &= \Phi_G[P_n - p_n] + \Phi_G\left[\sum_{k=0}^{n-1}(P_k - p_k)\frac{t^{n-k-1}}{(n-1-k)!}\right] \\ &= \Phi_G[P_n - p_n] + \Phi_{\theta_n}\left[\frac{t^{n-1}}{(t-1)!}\right]\end{aligned}$$

by the inductive assumption. Integrating $\Phi_G[P_n - p_n]$ by parts n times gives us

$$\Phi_G(P_n - p_n)\left[\frac{t^{n-1}}{(n-1)!}\right].$$

Now, combining the two integrals into one,

$$\Phi_G[P_n - p_n] + \Phi_{\theta_n}\left[\frac{t^{n-1}}{(t-1)!}\right] = \Phi_{\theta_n}\left[\frac{t^{n-1}}{(t-1)!}\right],$$

and the statement holds for $n+1$. Therefore the statement holds for all n. ■

This lemma is very useful for proving the following generalized linear differential comparison result.

Theorem 5.1.2. *Let p_j and P_j be continuous on $[0,b]$ for $0 \leq j \leq n-1$. Suppose ψ, $v \in C^n[0,b]$ with*

$$L(\psi) = \psi^{(n)} + \sum_{k=0}^{n-1} p_t \psi^{(k)} = 0$$

and initial conditions $\psi^{(k)} = 0$ for $0 \leq k \leq n-1$ and $\psi^{(n)} = 1$. Similarly, suppose

$$l(v) = v^{(n)} + \sum_{k=0}^{n-1} p_t v^{(k)} = 0$$

and initial conditions $v^{(k)} = 0$ for $0 \leq k \leq n-1$ and $v^{(n)} = 1$. Let $G(t,s)$ be the Green's function associated with L. If ψ is positive on $(0,b)$, $G(t,s)$ is a positive operator on Δ, and Θ_n is positive on Δ, then $v \geq \psi$ on $(0,b)$.

Proof. By Theorem 4.2.2, the statement holds if Φ_G, $I + R(\Theta_n)$, and

$$\left(I + \Phi_{R(\Theta_n)}\right) \circ \Phi_G \left[\sum_{k=0}^{n-1} (P_k - p_k) \frac{t^{n-k-1}}{(n-1-k)!} + \Phi_{K-M}(\psi^{(n-1)})\right]$$

are positive, where K and M are the Taylor kernels associated with l and L, respectively. The hypotheses of this theorem state that G and Θ_n are positive, so $I + R(\Theta_n)$ is positive. Now,

$$\Phi_G \left[\sum_{k=0}^{n-1} \frac{(P_k - p_k)t^{n-k-1}}{(n-1-k)!} + \Phi_{K-M}(\psi^{(n-1)})\right] = \Phi_G \left[\sum_{k=0}^{n-1} \frac{(P_k - p_k)t^{n-k-1}}{(n-1-k)!}\right] + \Phi_G \left[\Phi_{K-M}(\psi^{(n-1)})\right],$$

and by Lemma 5.1.1 and Lemma 4.2.1, this is equal to

$$\Phi_{\Theta_n} \left(\frac{t^{n-1}}{(t-1)!}\right) + \Phi_{\Theta_n * H}(\psi^{(n-1)}) = \Phi_{\Theta_n} \left[\frac{t^{n-1}}{(t-1)!} + \Phi_{H_n}(\psi^{(n-1)})\right],$$

which is positive since Φ_{Θ_n} and ψ are positive. Therefore

$$(I + \Phi_{R(\Theta_n)}) \circ \Phi_G \left[\sum_{k=0}^{n-1} (P_k - p_k) \frac{t^{n-k-1}}{(n-1-k)!} + \Phi_{K-M}(\psi^{(n-1)})\right]$$

is positive and $v \geq \psi$ on $(0, b)$ by Theorem 4.2.2.

■

This theorem will be used to prove the following n^{th}-order Sturm's theorem.

Theorem 5.1.3. *Let Q and q be continuous with $Q > q$ on $[0, b]$. Suppose ψ, $v \in C^n[0, b]$ with*

$$L(\psi) = \psi^{(n)} + Q\psi = 0,$$

and initial conditions $\psi^{(k)} = 0$ *for* $0 \leq k \leq n-1$ *and* $\psi^{(n)} = 1$*. Similarly, suppose*

$$l(v) = v^{(n)} + qv,$$

and initial conditions $v^{(k)} = 0$ *for* $0 \leq k \leq n-1$ *and* $v^{(n)} = 1$*. Let* $G(t,s)$ *be the Green's function associated with* L*. If* ψ *is positive on* $(0,b)$ *and* $G(t,s)$ *is a positive operator on* Δ*, then* $v \geq \psi$ *on* $(0,b)$*.*

Proof. In this case,

$$\theta_n = G(t,s)(Q(s) - q(s)),$$

which is a positive operator since G is positive and $Q > q$. Therefore, by the previous theorem, $v \geq \psi$. ■

5.2 Integral Versions of Sturm's Comparison Theorem

The next two results are integral versions of Sturm's Comparison Theorem.

Theorem 5.2.1. *Suppose* q*,* $Q \in C[0,b]$ *with* $Q(t) \geq q(t)$*,* $H(t,s)$ *is continuous on* $\Delta = \{(t,s) : 0 \leq s \leq t \leq b\}$*, and* $R \in C[\Delta]$ *is such that*

$$(I + \Phi_R) = (I + Q(t)\Phi_H)^{-1}.$$

Furthermore, suppose

$$\phi = -Q(t)[t + \Phi_H(\phi)]$$

and

$$u \geq -q(t)[t + \Phi_H(u)].$$

If $\Phi_H(\phi) + t > 0$*, then* $\Phi_H(u) \geq \Phi_H(\phi)$.

Proof. Suppose not. Then there exists $a \in (0, b)$ such that $\Phi_H(u) + t > 0$ throughout $(0, a]$ and

$$\Phi_H(u)(a) < \Phi_H(\phi)(a).$$

Note that in the interval $(0, a]$,

$$\begin{aligned} u(t) + Q(t)[t + \Phi_H(u)] &\geq -q(t)[t + \Phi_H(u)] + Q(t)[t + \Phi_H(u)] \\ &= [Q(t) - q(t)][t + \Phi_H(u)] \\ &\geq 0. \end{aligned}$$

Thus on the interval $(0, a]$,

$$\begin{aligned} \Phi_H(u(t) - \phi(t)) &= \Phi_H(I + \Phi_R)[I + Q(t)\Phi_H][u(t) - \phi(t)] \\ &= \Phi_H(I + \Phi_R)[(I + Q(t)\Phi_H)(u) + Q(t)t] \\ &\geq \Phi_H(I + \Phi_R)[u(t) + Q(t)(t + \Phi_H(u))] \\ &\geq 0. \end{aligned}$$

The last inequality follows by 4.2.1, which states $G = H + H * R$ is the Green's function for the equivalent differential equation. In this case, we can say $G > 0$

on Δ. Thus we have a contradiction since we assumed $\Phi_H(u)(a) < \Phi_H(\phi)(a)$. Therefore, if $\Phi_H(\phi) + t > 0$, then $\Phi_H(u) \geq \Phi_H(\phi)$. ■

Theorem 5.2.2. *Let M, K, and R be continuous on $\Delta = \{(t,s) : 0 \leq s \leq t \leq 0\}$ with*

$$(I + \Phi_R) = (I - \Phi_M)^{-1}.$$

Define $G = H + H * R$, where H is as before: $H(t,s) = (t-s)$. Let ϕ, u, A, B, and $C \in C[0,b]$ with $A(t) \geq 0$, $B(t) \geq C(t)$,

$$\phi(t) = C(t) + \Phi_M(\phi)(t),$$

$$u(t) \geq B(t) + \Phi_K(u)(t),$$

and

$$\Phi_{G*[K-M]}(u) = \Phi_{[A(s)G]*H}(u).$$

If $\Phi_H(\phi) > 0$, then $\Phi_H(u) \geq \Phi_H(\phi)$.

Proof. Suppose not. Then there exists an $a \in [0,b]$ such that

$$\Phi_H(u)(a) < \Phi_H(\phi)(a)$$

with $\Phi_H(u) > 0$ on $(0, a]$. Thus, on $(0, a]$, we have

$$\begin{aligned}
\Phi_H(u - \phi) &= \Phi_G[\Phi_{I-M}(u - \phi)] \\
&= \Phi_G[u - \Phi_M(u) - \phi + \Phi_M(\phi)] \\
&= \Phi_G[u - \Phi_M(u) - C] \\
&\geq \Phi_G[B - C + \Phi_{K-M}(u)] \\
&\geq \Phi_{G*[K-M]}(u) \\
&= \Phi_{[A(s)G]*H}(u) \\
&> 0,
\end{aligned}$$

which is a contradiction. Therefore, if $\Phi_H(\phi) > 0$, then $\Phi_H(u) \geq \Phi_H(\phi)$ on $[0, b]$.

■

CHAPTER 6

OSCILLATION OF COEFFICIENTS IN SECOND-ORDER DIFFERENTIAL EQUATIONS

6.1 Introduction to a Theorem

For $1 < p < \infty$, consider the zeros in $(0, \infty)$ of solutions of

$$u'' + qu = 0, \quad (u, u')(0) = (0, 1),$$

as q ranges over the nonnegative, continuous functions in $[0, \infty)$ such that $\|q\|_p = 1$. We wish to determine the infimum $\beta(p)$ of those zeros, show that it is a minimum exactly when $p > 1$, and describe q and u in those cases.

Motivation: In general, we are looking at solutions to $u'' + qu = 0$ in $[0, \infty)$, with $(u, u')(0) = (0, 1)$, $q \geq 0$, and $\|q\|_p = 1$. Specifically, what is the smallest possible zero for u?

Given a solution, we know

$$u(t) = t + \Phi_k(u)(t)$$

or

$$(I - \Phi_K)(u) = id,$$

where id represents the identity function. Suppose we have $\beta \in [0, \infty)$. Without loss of generality, assume $u(\beta) > 0$. Can we change q slightly to reduce $u(\beta)$ while still maintaining $\|q\|_p = 1$? Replace q with $q + \epsilon\hat{q}$. Then we have

$$(I - \Phi_{q+\epsilon\hat{q}})(u_{q+\epsilon\hat{q}}) = id.$$

Taking the derivative with respect to ϵ, we get

$$-\Phi_{\widehat{K}} u + (I - \Phi_K)\hat{u} = 0,$$

where $\hat{u}$ is the rate of change of the solution with respect to ϵ. That is,

$$\hat{u}(t) = \left(I + \Phi_{R(K)}\right)\Phi_{\widehat{K}} u(t) = \int_0^t -(t-s)\hat{q}(s)u(s)ds + \int_0^t R(t,s)\left[\int_0^s -(s-r)\hat{q}(r)u(r)dr\right] ds.$$

Can we choose $\hat{q}$ so that $\hat{u}(\beta) = 0$? We require that $\|q + \epsilon\hat{q}\|_p = 1$, forcing the restriction

$$\int_0^\infty pq^{p-1}\hat{q} = 0.$$

If we switch the order of integration and perform a simple algebraic manipulation on $\hat{u}$, we see that

$$\hat{u}(t) = \int_0^t -\hat{q}(s)u(s)\left[(t-s) + \int_s^t -R(t,s)(r-s)dr\right] ds = \int_0^t -\hat{q}(s)u(s)G(t,s)ds$$

by Proposition 4.2.1, where $G(t,s)$ is the Green's function associated with our differential equation. Thus

$$\hat{u}(\beta) = \int_0^\beta -\hat{q}(s)u(s)G(\beta,s)ds.$$

We will call q a critical choice if $\hat{u}(\beta) = 0$ for all such $\hat{q}$. Notice that the only time

$$\int_0^\beta pq^{p-1}\hat{q} = 0$$

and $\hat{u}(\beta) = 0$ is when $u(t)G(\beta, t)$ is a constant multiple of $pq^{p-1}(t)$. Thus q^{p-1} must be equal to the product of two solutions, one with a zero at 0 and with a zero at β. Therefore, it must be that $q^{(p-1)/2}$ solves the equation.

If $r = q^{(p-1)/2}$, then we have

$$r'' + qr = 0$$

or

$$r'' + r^{(p+1)/(p-1)} = 0.$$

The goal is to find the first zero of r, $\beta(p)$. To make calculations more straightforward, we shift things to be symmetric around the origin. That is, find r such that

$$r'' + r^{(p+1)/(p-1)} = 0$$

and

$$(r, r')(0) = (1, 0).$$

The goal is to find the first positive zero, $f(p)$, (and hence the first negative zero $-f(p)$) of r. We will find $f(p)$ and then manipulate r and q so that they satisfy the original differential equation and ultimately find $\beta(p)$.

The first integral

$$(1/2)(r')^2 + (p-1)/(2p)r^{2p/(p-1)}$$

is constant along r ([12], pp. 16-18), in particular at $t = 0$. Thus

$$(1/2)(r')^2 + (p-1)/(2p)r^{2p/(p-1)} = (p-1)/(2p).$$

Solving for r' results in

$$dr/dt = [(p-1)/p(1 - r^{2p/(p-1)})]^{1/2}$$

and hence

$$dt/dr = [(p-1)/p(1 - r^{2p/(p-1)})]^{-1/2}.$$

To find $f(p)$, integrate from 0 to 1:

$$f(p) = \int_0^1 [(p-1)/p(1 - s^{2p/(p-1)})]^{-1/2} ds = (p/(p-1))^{1/2} \int_0^1 (1 - s^{2p/(p-1)})^{-1/2} ds.$$

Using the substitution $w = s^{2p/(p-1)}$, we get

$$f(p) = (1/2)(p/(p-1))^{1/2}((p-1)/p)^{1/2} \int_0^1 w^{(-p-1)/(2p)}(1-w)^{1/2} dw.$$

Finally,

$$f(p) = (1/2)((p-1)/p)^{1/2} B((p-1)/(2p), (1/2)),$$

where B is the beta function defined by

$$\begin{aligned} B(x,y) &= \int_0^1 t^{x-1}(1-t)^{y-1}\, dt, \qquad x, y > 0 \\ &= \frac{\Gamma(x)\Gamma(y)}{\Gamma(x+y)} \end{aligned}$$

and

$$\Gamma(x) = \int_0^\infty t^{x-1} e^{-t}\, dt.$$

Notice in this symmetric case, we have not yet required $\|q\|_p = 1$, so we will address this now. First, we compute $\int_{-f(p)}^{f(p)} q^p$. By using the first integral and integrating by parts, we get the following result:

$$\begin{aligned} \int_{-f(p)}^{f(p)} q^p(t)\, dt &= \int_{-f(p)}^{f(p)} r^{\frac{2p}{p-1}}(t)\, dt && (*) \\ &= \int_{-f(p)}^{f(p)} 1 - \left(\frac{p}{p-1}\right)(r')^2(t)\, dt \\ &= 2f(p) - \frac{p}{p-1} \int_{-f(p)}^{f(p)} (r')^2(t)\, dt \\ &= 2f(p) - \frac{p}{p-1} \int_{-f(p)}^{f(p)} r^{\frac{2p}{p-1}}(t)\, dt. && (**) \end{aligned}$$

Setting $(*) = (**)$ and solving for

$$\int_{-f(p)}^{f(p)} r^{2p/(p-1)}(t)\, dt,$$

we get

$$\int_{-f(p)}^{f(p)} r^{(2p)/(p-1)}(t)\, dt = 2(p-1)f(p)/(2p-1). \tag{6.1}$$

We need to manipulate r into some function v so that the differential equation is still satisfied and

$$\|v^{2p/(p-1)}\|_p = 1.$$

Let

$$v(t) = \left[\frac{2f(p)(p-1)}{2p-1}\right]^{\frac{-p+1}{2p-1}} r\left(\left[\frac{2f(p)(p-1)}{2p-1}\right]^{\frac{-1}{2p-1}} t\right). \tag{6.2}$$

Notice that the zeros of v are then

$$f_v(p) := f(p)\left[\frac{2f(p)(p-1)}{2p-1}\right]^{\frac{1}{2p-1}}, \tag{6.3}$$

which can be simplified algebraically and by using properties of the beta function to

$$f_v(p) = \frac{1}{2}p^{\frac{-p}{2p-1}}\,(p-1)^{\frac{-p+1}{2p-1}}\,(2p-1)B\left(\frac{3p-1}{2p}, \frac{1}{2}\right)^{\frac{2p}{2p-1}}.$$

Then

$$\int_{-f_v(p)}^{f_v(p)} v^{2p/(p-1)}(t)\,dt = \int_{-f_v(p)}^{f_v(p)} [2f(p)(p-1)/(2p-1)]^{(-p+1)/(2p-1)} r([2f(p)(p-1)/(2p-1)]^{-1/(2p-1)}t)\,dt.$$

Using the change of variable

$$s = [2f(p)(p-1)/(2p-1)]^{-1/(2p-1)}t,$$

the integral becomes

$$(2p-1)/[2f(p)(p-1)] \int_{-f(p)}^{f(p)} r^{2p/(p-1)}(s)\,ds = 1$$

by (6.1).

We observe v also satisfies the differential equation since

$$v''(t) + v^{\frac{p+1}{p-1}}(t) = \left[\frac{2f(p)(p-1)}{2p-1}\right]^{\frac{-p-1}{2p-1}} (r(s) + r''(s)) = 0,$$

where s is the same as in the change of variable above.

If we shift the symmetric case to the origin, our solution (6.2) remains the same only shifted by $f_v(p)$, and this will shift the first possible zero to twice (6.3), resulting in the following theorem:

Theorem 6.1.1. *Let*

$$\beta(p) = p^{\frac{-p}{2p-1}} (p-1)^{\frac{-p+1}{2p-1}} (2p-1) B\left(\tfrac{3p-1}{2p}, \tfrac{1}{2}\right)^{\frac{2p}{2p-1}},$$

where

$$B(x,y) = \int_0^1 t^{x-1}(1-t)^{y-1}\,dt,$$

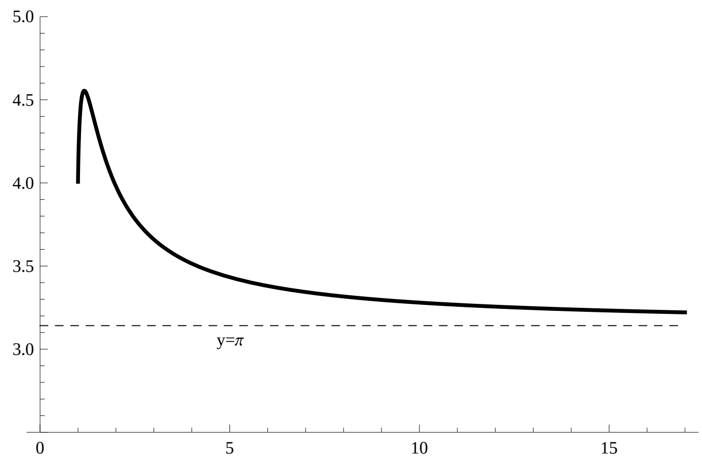
the beta function. Let u_q be the solution to the initial value problem

$$u'' + qu = 0\,, \qquad (u,u')(0) = (0,1), \tag{6.4}$$

where $q \in \mathcal{C}[0,b]$ such that the L^p-norm $\|q\|_p \leq 1$. Then

(a) $\beta(p) = \inf\{z \in (0,b] : u_q(z) = 0$ *for some* $q\}$

(b) Given $p > 1$ and $b \geq \beta(p)$, there is a unique function $q \in \mathcal{C}[0,b]$ such that the solution u_q to (6.4) vanishes at $\beta(p)$. Moreover, $q > 0$ on $(0, \beta(p))$ and $q = 0$ almost everywhere on the complement $[0,b] \setminus (0,\beta(p))$.



The discussion in the chapter leading up to Theorem 6.1.1 has shown (a). The second part of Theorem 6.1.1 will require a number of preliminary results contained in the following section.

6.2 Lemmas Needed to Prove Theorem 6.1.1

Throughout this chapter,

$$\left(\Phi_q u\right)(t) = \int_0^t (t-s)q(s)u(s)\,ds$$

and

$$u_q = \left(I - \Phi - q\right)^{-1}(id)$$

for $q \in L^p[0,b]$, $u \in C[0,b]$ and $t \in [0,b]$.

Lemma 6.2.1. *For $q \in L^p[0,b]$, $u \in C[0,b]$, and $t \in [0,b]$,*

$$\left(\Phi_q u\right)(t) = \int_0^t (t-s)q(s)u(s)\,ds$$

converges.

Proof. Since $q \in L^p[0,b]$, Hölder's inequality gives

$$||q||_1 \leq b^{(p-1)/p}||q||_p,$$

which implies that

$$\left|\left(\Phi_q u\right)(t)\right| \leq t||q||_1 \max_{t\in[0,b]}|u|.$$

Thus $\Phi_q u$ converges. ∎

Lemma 6.2.2. *Each function $\Phi_q u$ is C^1.*

Proof. First, a straightforward epsilon-delta proof will give us continuity. Given $a \in [0, b]$, for all $\epsilon > 0$, let

$$\delta = \frac{\epsilon}{2||q||_1 \max|q|}.$$

Then, if $|t - a| < \delta$, we have

$$\begin{aligned}
|(\Phi_q u)(t) - (\Phi_q u)(a)| &= \left| -\int_0^a (a-s)q(s)u(s)\,ds + \int_0^t (t-s)q(s)u(s)\,ds \right| \\
&= \left| \int_0^a (t-a)q(s)u(s)\,ds + \int_a^t (t-s)q(s)u(s)\,ds \right| \\
&\leq \left| \int_0^a (t-a)q(s)u(s)\,ds \right| + \left| \int_a^t (t-a)q(s)u(s)\,ds \right| \\
&\leq |t-a| \cdot ||q||_1 \max|u| + |t-a| \cdot ||q||_1 \max|u| \\
&\leq 2|t-a| \cdot ||q||_1 \max|u| \\
&\leq 2\delta ||q||_1 \max|u| \\
&= \epsilon.
\end{aligned}$$

Now we turn to $(\Phi_q u)'$.

$$\begin{aligned}
(\Phi_q u)'(t) &= \left(-\int_0^t (t-s)q(s)u(s)\,ds \right)' \\
&= \left(-t\int_0^t q(s)u(s)\,ds \right)' + \left(\int_0^t sq(s)u(s)\,ds \right)' \\
&= -\int_0^t q(s)u(s)\,ds - tq(t)u(t) + tq(t)u(t) \\
&= -\int_0^t q(s)u(s)\,ds.
\end{aligned}$$

Because q, u, and integration are all continuous, we have that $(\Phi_q u)'$ must be continuous. ■

Lemma 6.2.3. *The uniform bounds*

$$\text{(a) } ||\left(\Phi_q u\right)||_\infty \le C||q||_p||u||_\infty$$

and

$$\text{(b) } ||\left(\Phi_q u\right)'||_\infty \le D||q||_p||u||_\infty$$

hold, and, in particular, Φ_q *is a bounded linear endomorphism of* $C[0,b]$.

Proof. We start by proving (b):

$$||\left(\Phi_q u\right)'||_\infty \le \int_0^b |q(s)u(s)|\,ds \le ||q||_1||u||_\infty \le b^{(p-1)/p}||q||_p||u||_\infty,$$

where the last inequality was previously established by Hölder's inequality.

(a) By the mean value theorem, for any $t \in (0,b]$ there exists $c \in (0,t)$ such that

$$\frac{\left(\Phi_q u\right)(t) - \left(\Phi_q u\right)(0)}{t} = \left(\Phi_q u\right)'(c).$$

Observing that $\left(\Phi_q u\right)(0) = 0$ and multiplying both sides by t, we obtain

$$\left(\Phi_q u\right)(t) = t\left(\Phi_q u\right)'(c) \le b\left(\Phi_q u\right)'(c).$$

Taking the norm, we have

$$||\Phi_q u||_\infty \le b||\left(\Phi_q u\right)'||_\infty \le b^{(2p-1)/p}||q||_p||u||_\infty.$$

■

Lemma 6.2.4. *The series $\sum_{n=0}^{\infty} \Phi_q^n$ converges in the space of bounded linear operators and therefore produces a two-sided inverse $I - \Phi_q$.*

Proof. We claim that

$$|| \left(\Phi_q^n u\right)(t)|| \leq ||q||_1^n ||u||_\infty \left(\frac{t^n}{n!}\right).$$

We prove the claim by induction.

$\underline{n=0}$: $0 \leq ||q||_1 ||u||_\infty$.

Now suppose it holds for some n. Then

$$\begin{aligned}
\left|\left(\Phi_q^{n+1} u\right)(t)\right| &\leq \left|-\int_0^t (t-s) q(s) ||q||_1^n ||u||_\infty \frac{s^n}{n!}\, ds\right| \\
&= \frac{||q||_1^n ||u||_\infty}{n!} \left|-\int_0^t (t-s) q(s) s^n \, ds\right| \\
&\leq \frac{||q||_1^n ||u||_\infty}{n!} \int_0^t |q(s)(t-s)s^n| \, ds \\
&\leq \frac{||q||_1^n ||u||_\infty}{n!} \cdot ||q||_1 \max_{s\in[0,t]} \left\{(t-s)s^n\right\}.
\end{aligned}$$

The last inequality follows by applying Lemma 6.2.3. We now find the maximum of $(t-s)s^n$ by first setting the derivative of $ts^n - s^{n+1}$ equal to zero and solving for s:

$$\begin{aligned}
nts^{n-1} - (n+1)s^n = 0 \\
nts^{n-1} &= (n+1)s^n \\
nt &= (n+1)s^n \\
\frac{nt}{n+1} &= s^n.
\end{aligned}$$

We observe that s is definitely in the interval $[0, t)$. Therefore $\max(t-s)s^n$ is

$$\left(t - \frac{nt}{n+1}\right)\left(\frac{nt}{n+1}\right)^n = \frac{t}{n+1} \cdot t^n \cdot \left(\frac{n}{n+1}\right)^n \leq \frac{t^{n+1}}{n+1}.$$

Thus

$$\left|\left(\Phi_q^n u\right)(t)\right| \leq ||q||_1^n \|u\|_\infty \left(t^n/n!\right)$$

for all n, which implies that

$$||\Phi_q^n|| \leq ||q||_1^n \cdot \left(b^n/n!\right).$$

■

Lemma 6.2.5. *The series* $\sum_{n=0}^{\infty} \Phi_q^n$ *converges in the space of bounded linear endomorphisms and therefore produces a two-sided inverse* $I - \Phi_q$.

Proof. From Lemma 6.2.1, we have

$$\sum_{n=0}^{\infty} \Phi_q^n \leq \sum_{n=0}^{\infty} \|\Phi_q^n\| \leq \sum_{n=0}^{\infty} \frac{\left(\|q\|_1 b\right)^n}{n!} = \exp\left(\|q\|_1 b\right) \leq \exp\left(b^{(2p-1/p}\|q\|_p\right),$$

using the Taylor series for e and Lemma 6.2.3. Thus, by Theorem 2.1.4, $I - \Phi_q^n$ is invertible and

$$\left(I - \Phi_q^n\right)^{-1} = \sum_{n=0}^{\infty} \Phi_q^n$$

because the space is complete. ■

Lemma 6.2.6. *There is a uniform bound on the operator norms* $\| (I - \Phi_q)^{-1} \|$ *for* $q \in L^p[0,b]$ *with* $\|q\|_p \leq 1$.

Proof. For $q \in L^p[0,b]$, we have

$$\| (I - \Phi_q)^{-1} \| = \left\| \sum \Phi_q^n \right\| \leq \sum \|\Phi_q^n\| \leq \exp\left(b^{(2p-1)/p} \|q\|_p\right).$$

■

Lemma 6.2.7. *Let* $q \in L^p[0,b]$ *and* $a \in (0,b]$. *If* $Q \in L^p[0,b]$ *with* $Q \geq q$ *and* u_Q *has no zero in* $(0,a)$, *then* u_q *has no zero in* $(0,a)$. *Furthermore,* u_q *has no zero in* $(0,a]$ *unless* $u_Q(a) = 0$, *in which case* $q = Q$ *almost everywhere in* $[0,a]$.

Proof. We claim that

$$\left(u_Q u_q' - u_Q' u_q\right)(t) = \int_0^t (Q-q)(s) u_Q(s) u_q(s)\, ds$$

for $t \in [0,b]$. To prove this, we take the derivative of the left-hand side:

$$\begin{aligned}
\left(u_Q u_q' - u_Q' u_q\right)'(t) &= \left[u_Q' u_q' + u_Q u_q'' - u_Q'' u_q - u_Q' u_q'\right](t) \\
&= \left[u_Q u_q'' - u_Q'' u_q\right](t) \\
&= \left[(Q-q)\, u_Q u_q\right](t).
\end{aligned}$$

We observe that for fixed t, both mappings

$$q, Q \mapsto \left(u_Q u_q' - u_Q' u_q\right)'(t)$$

and

$$q, Q \mapsto \int_0^t (Q-q)(s)u_Q(s)u_q(s)\,ds$$

are continuous by Hölder's inequality and Lemma 6.2.3, so the claim holds.

Now suppose u_q has a zero in $(0,a]$ and let z be the least such zero. Then we have

$$\left(u_Q u_q' - u_Q' u_q\right)'(z) = \int_0^z (Q-q)(s)u_Q(s)u_q(s)\,ds,$$

or

$$u_Q(z)u_q'(z) = \int_0^z (Q-q)(s)u_Q(s)u_q(s)\,ds.$$

We note that the left-hand side of the last equation is less than or equal to zero, while the right-hand side is nonnegative because both u_q and u_Q are positive on $[0,z]$. This forces both sides to be zero, and hence $q = Q$ almost everywhere in $[0,z]$, which also forces $z = a$. ■

6.3 A Minimum Exists

Finally, we need to show that the set

$$\{z \in (0,b] : u_q(z) = 0, q \in L^p[0,b], \|q\|_p \leq 1\}$$

actually attains a minimum. We first note that for sufficiently large values of b, the set is non-empty. For example, let

$$b > \left(\frac{\pi}{2}\right)^{2p/(2p-1)}$$

and take

$$q(0) = \left(\frac{\pi}{2b}\right)^2.$$

Then

$$u_q(t) = \frac{2b}{\pi} \cos\left(\frac{\pi t}{2b}\right)$$

satisfies $u'' + qu = 0$, $\|q\|_p < 1$, and the first zero occurs at $z = b$.

Lemma 6.3.1. *The infimum*

$$\beta = \inf\left\{z \in (0, b] : u_q(z) = 0, q \in L^p[0, b], \|q\|_p \leq 1\right\}$$

is positive.

Proof. First we observe that

$$u_q(t) = \left(I - \Phi_q\right)^{-1}(id)(t),$$

implying that

$$t = \left(\left(I - \Phi_q\right) u_q\right)(t) = u_q(t) = \int_0^t (t - s)q(s)u(s)\, ds.$$

By Hölder's inequality and Lemma 6.2.3, we have

$$\| \left(\Phi_q u_q\right)' \|_\infty \leq b^{(p-1)/p} \|q\|_p \|u\|_\infty \leq b^{(2p-1)/p} \exp\left(b^{(2p-1)/p}\right).$$

Because

$$\left(\Phi - qu\right)(0) = 0,$$

we have

$$\begin{aligned} u_q(t) &= t - \int_0^t (t-s) q(s) u_q(s)\, ds \\ &\geq t - \max_{s \in [0,t]} \left|(t-s) u_q(s)\right| \cdot \int_0^t \left|q(s)\right|\, ds \\ &\geq t - t^2 \exp\left(t^{(p-1)/p}\right) t^{(p-1)/(p-1)} \\ &= t \cdot \left(1 - t^{(2p-1)/p} \exp\left(t^{(2p-1)/p}\right)\right). \end{aligned}$$

If we solve

$$1 - t^{(2p-1)/p} \exp\left(t^{(2p-1)/p}\right) > 0,$$

we find

$$t < (W(1))^{p/(2p-1)} \approx (.567143)^{p/(2p-1)},$$

where W is the Lambart W-function defined by $W(x)e^{W(x)} = x$ for $x \in [-e^{-1}, \infty]$ [14]. It follows that

$$\beta > (W(1))^{p/(2p-1)}.$$

$\blacksquare$

We note that as p gets very large,

$$(W(1))^{p/(2p-1)} \to \sqrt{W(1)} \approx .753089.$$

Lemma 6.3.2. *There exists $q \in L^p[0,b], \|q\|_p \leq 1,$ such that $u_q(\beta) = 0$.*

Proof. Choose a sequence $\{q_j\}_{j\to\infty}$ in the closed unit ball of L^p such that the u_{q_j} have zeros $z_j > 0$ with

$$\lim_{j=1}^{\infty} z_j = \beta.$$

Again we have the following relation: $u_{q_j} = id + \Phi_{q_j}\left(u_{q_j}\right)$. By Lemma 6.2.3, the $\Phi_{q_j}\left(u_{q_j}\right)$ are uniformly bounded with uniformly bounded derivatives. Utilizing the Arzela-Ascoli Theorem, we can choose a subsequence which converges uniformly to some $u \in C[0,b]$. Because the closed unit ball of L^p is sequentially compact in the weak topology, we can again choose a subsequence so that for all $f \in L^{p/(p-1)}[0,b]$,

$$\lim_{j\to\infty} \int_0^b q_j(s)f(s)\,ds = \int_0^b q(s)f(s)\,ds$$

for some $q \in L^p[0,b]$, with $\|q\|_p \leq 1$. The resulting subsequence $\{\Phi_{q_j} u_{q_j}\}_{j=1}^{\infty}$ converges pointwise to $\Phi_q u$, and since the $\left(\Phi_{q_j} u\right)'$ are uniformly bounded, the convergence is uniform. Thus we have

$$u = \lim_{j\to\infty} u_{q_j} = \lim_{j\to\infty} \left(id + \Phi_{q_j} u_j\right) = \lim_{j\to\infty} \left(id + \Phi_{q_j} u\right) = id + \Phi_q u.$$

Therefore $u = u^q$ and $u(\beta) = 0$. ■

Lemma 6.3.3. *Suppose $q \in L^p[0,b]$, with $\|q\|_p \leq 1$ and $u_q(\beta) = 0$. Then $q \geq 0$ and $\|q\|_p = 1$.*

Proof. Let

$$Q = |q| / \left(\int_0^b |q(s)|^p \, ds \right)^{1/p}$$

on $[0,b]$ and zero elsewhere. Then $Q \geq q$ and $\|Q\|_p = 1$ by construction. By Lemma 6.2.7, $q = Q$ almost everywhere on $[0,b]$, and the requirement $\|q\|_p \leq 1$ implies that $q = 0$ outside of $[0,b]$. ■

Let q_β be the minimizing q we found in Lemma 6.3.2. By Lemma 6.3.3, q_β satisfies the conditions we placed on q in the introduction of this chapter, which led to the discovery of $\beta(p)$. Therefore $\beta(p)$ is the minimum zero and the minimizing solution u_{q_β} is a multiple of $pq^{p-1}(t)$, as we also found in the discussion. This proves Theorem 6.1.1.

6.4 Further Results

Corollary 6.4.1. *Suppose* $1 \leq p \leq \infty$*. Given* $u'' + qu = 0$*,* $(u, u')(0) = (0, 1)$ *and* $||q||_p = m$*, the infimum of the first possible zero of solutions is given by*

$$\beta_m(p) = m^{\frac{-p}{2p-1}} \beta(p).$$

Proof. If $||q||_p = m$, then in the discussion above, (6.2) becomes

$$v(t) = \left[\frac{2f(p)(p-1)}{m^p(2p-1)}\right]^{\frac{-p+1}{2p-1}} r\left(\left[\frac{2f(p)(p-1)}{m^p(2p-1)}\right]^{\frac{-1}{2p-1}} t\right),$$

which implies that the smallest possible zero changes by a factor of $m^{-p/(2p-1)}$. Thus, the new zero is

$$\beta_m(p) := m^{-p/(2p-1)} \beta(p).$$

■

Corollary 6.4.2. *On* $[0, b]$*, suppose we have the equation*

$$u'' + qu = 0, \quad (u, u')(0) = (0, 1),$$

with $q \in C[0, b]$*,* $q \geq 0$*, and* $||q||_p = m$*. Then the number of zeros* n *of a solution in* $[0, b]$ *is bounded:*

$$n \leq \left(\frac{b}{\beta(p)}\right)^{\frac{2p-1}{2p}} m^{1/2}.$$

Proof. Suppose that u is a solution and that it has n zeros in $[0, b]$. Without loss of generality, assume the zeros are evenly distributed; that is, they occur at $b/n, 2b/n, 3b/n, \ldots, (n-1)b/n, b$. Let

$$k_i = \left[\int_{\frac{(i-1)n}{b}}^{\frac{ib}{n}} q^p(s)ds \right]^{1/p}, \ i = 1, \ldots, n.$$

Then

$$b/n \geq \beta(p) k_i^{-p/(2p-1)}$$

by Corollary 6.4.1, and

$$k_i^{p/(2p-1)} \geq n\beta(p)/b,$$

which implies that

$$k_i^p \geq [n\beta(p)/b]^{2p-1}$$

for $i = 1, \ldots, n$. Now, since $||q||_p = m$, we have

$$\begin{aligned} m^p &= \int_0^b q^p(s)ds \\ &= \sum_{i=1}^{n} \int_{\frac{(i-1)n}{b}}^{\frac{ib}{n}} q^p(s)ds \\ &= \sum_{i=1}^{n} k_i^p \\ &\geq n \left[\frac{n\beta(p)}{b}\right]^{2p-1} \\ &= n^{2p} \left[\frac{\beta(p)}{b}\right]^{2p-1}. \end{aligned}$$

Solving the inequality

$$m^p \geq n^{2p} \left[\beta(p)/b\right]^{2p-1}$$

for n, we find

$$(b/\beta(p))^{(2p-1)/(2p)} m^{1/2} \geq n.$$

$\blacksquare$

Notice that if we look at $p = \infty$, we get $n \leq (b/\pi)\sqrt{m}$.

Let us now consider $0 < p < 1$. We claim that, given an interval $[0, b]$ and values n and ε, we can construct a function q so that

$$\int |q|^p \leq \varepsilon$$

and a solution to the differential equation

$$u'' + qu = 0, \quad (u, u')(0) = (0, 1),$$

has n zeros in $[0, b]$.

It is enough to show the construction on the first $[0, b/n]$ since the process is identical for the other $n-1$ intervals. That is, we can construct q so that the solution has a zero in $[0, b/n]$ while keeping $\int_0^{b/n} |q|^p$ as small as we like.

Let $\eta > 0$ and define

$$\widehat{q}(t) := \frac{\frac{\pi}{2\eta}\sin\left(\frac{\pi}{2\eta}\left(t - \frac{b}{2n} + \eta\right)\right)}{\frac{2\eta}{\pi}\sin\left(\frac{\pi}{2\eta}\left(t - \frac{b}{2n} + \eta\right)\right) + \frac{b}{2n} - \eta}.$$

Notice that $\widehat{q}\left(\frac{b}{2n} - \eta\right) = 0$ and $\widehat{q}\left(b/(2n) + \eta\right) = 0$, which implies that

$$q(t) := \begin{cases} 0 & \text{if } \ 0 \le t < \frac{b}{2n} - \eta \\ \\ \widehat{q}(t) & \text{if } \ \frac{b}{2n} - \eta \le t \le \frac{b}{2n} + \eta \\ \\ 0 & \text{if } \ \frac{b}{2n} + \eta < t < \frac{b}{n} \end{cases}$$

is a continuous function.

We also have

$$q\left(b/(2n)\right) = \pi^2 n/(4n\eta^2 + b\pi\eta - 2n\eta^2\pi),$$

so

$$\lim_{\eta \to 0} q\left(b/(2n)\right) = \infty,$$

implying that q behaves like a delta function at $b/(2n)$ as η gets very small. Thus, we can make $||q||_p < \varepsilon$ by picking η sufficiently small.

Now, let

$$\widehat{u}(t) := \frac{2\eta}{pi} \sin\left(\frac{\pi}{2\eta}\left(t - b/(2n) + \eta\right)\right) + b/(2n) - \eta.$$

We observe that

$$\widehat{u}\left(b/(2n) - \eta\right) = \widehat{u}\left(b/(2n) + \eta\right) = b/(2n) - \eta,$$

so the function defined by

$$u(t) := \begin{cases} t & \text{if} \quad 0 \leq t < b/(2n) - \eta \\ \\ \widehat{u}(t) & \text{if} \quad b/(2n) - \eta \leq t \leq b/(2n) + \eta \\ \\ -t + \frac{b}{n} & \text{if} \quad b/(2n) + \eta < t \leq \frac{b}{n} \end{cases}$$

is continuous with a zero at $t = b/n$.

Since $u(t)$ also satisfies $u'' + qu = 0$, $(u, u')(0) = (0, 1)$, we have constructed the u and q that we needed.

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