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Metric nilpotent Lie algebras defined by graphs

Matthew Schroeder

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To the Graduate Faculty:

The members of the committee appointed to examine the thesis of Matthew Schroeder find it satisfactory and recommend that it be accepted.

Dr. Tracy Payne,
Major Adviser

Dr. Yu Chen,
Committee Member

Dr. Cathy Kriloff,
Committee Member

Dr. Shannon Kobs-Nawotniak,
Graduate Faculty Representative

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Abstract

In this thesis, we study classes of two-step metric nilpotent Lie algebras: uniform type Lie algebras and Heisenberg-like Lie algebras. In both classes we define a graph relation on basis elements to encode an algebra as an edge colored directed graph. We prove general properties and in the uniform type case, we classify the associated graphs in low dimensions. We present new examples of both kinds of metric Lie algebras, including infinite families of examples of both types. Finally we provide a graph-based method for combining Heisenberg-like Lie algebras or uniform Lie algebras to construct new Heisenberg-like Lie algebras or uniform type Lie algebras.

Chapter 1

Introduction

When studying Lie algebras, it is common to focus on a subclass of Lie algebras which contain extra structure or symmetry. The Levi Decomposition states each finite dimensional Lie algebra is the semi-direct product of a solvable Lie algebra with a semi-simple Lie algebra. Cartan and Killing provide a complete classification of semi-simple Lie algebras. Because of this, the structure of finite dimensional Lie algebras has been reduced to solvable Lie algebras. We explore a subclass of solvable Lie algebras, nilpotent Lie algebras, which are interesting due to applications in mathematics and physics such as the Heisenberg Lie algebra and examples in spectral theory.

In his thesis, Umlauf, presented a classification of non-trival nilpotent Lie algebras of dimension six and less over $\mathbb{C}$ [23]. As with many classifications of nilpotent Lie algebras, Umlauf's list was incomplete and contained errors. In 1958, Morozov provided a classification of nilpotent Lie algebras up to dimension 6 over fields of characteristic 0 [19]. Vergne, in her 1966 thesis, provided a classification of nilpotent Lie algebras of dimension 6 and less over $\mathbb{R}$ and $\mathbb{C}$ [24]. There have been several attempts using various methods to classify nilpotent Lie algebras up to dimension seven. In particular, Ancochea and Goze provide a classification of nilpotent Lie algebras up to dimension 7 over $\mathbb{C}$ [1], however they published a later paper [11] fixing errors in their previous classification, but it still contained errors. Seeley, in his thesis, provides an almost perfect classification of complex nilpotent Lie algebras over dimension 7 [22]. In 1998, Gong provides a widely accepted classification of nilpotent Lie algebras up to dimension 7 over $\mathbb{R}$ [9]. Later, in 2010, Magnin, developed a new approach for classifying complex Lie algebras and produced a classification which agreed with Gong's classification over dimension 7 [16]. A survey of the classification of nilpotent Lie algebras can be found in [12] and a historical survey can be found in [3].

There are a number of subclasses of nilpotent Lie algebras which have been of interest such as free nilpotent Lie algebras, complex two-step nilpotent Lie algebras (classified up to dimension 9) [8], and filiform Lie algebras (classified up to dimension 11) [12, p. 84].

Semi-simple Lie algebras have been previously classified through Dynkin diagrams [13]. Dynkin diagrams are a type of graph, and graphs have since been used to construct examples of two-step nilpotent Lie algebras. In particular, Dani and Mainkar defined a class of two-step nilpotent Lie algebras by simple uncolored undirected graphs [4]. Mainkar later proved two such Lie algebras are isomorphic if and only if their underlying graphs are isomorphic [18]. Ray defined Lie algebras by Schreier graphs which are directed and colored [21]. In this thesis, we construct edge colored directed graphs to encode basis elements of nilpotent Lie algebras. Specifically, we focus on two classes of nilpotent Lie algebras: a class of nilpotent Lie algebras called uniform type Lie algebras and a class of real metric nilpotent Lie algebras called Heisenberg-like Lie algebras.

Uniform type Lie algebras are defined by combinatorial properties and are highly symmetric. These algebras were first defined by DeLoff. His original motivation for defining this class was the study of Einstein solvmanifolds [6]. Payne analyzed soliton metrics on deformations of uniform type Lie algebras [20]. We focus on these Lie algebras over an arbitrary field $\mathbb{F}$. Due to the combinatorial definition of uniform type Lie algebras, there is a natural graph bijection between uniform Lie algebras and a class of "uniform graphs". We give a partial classification of all such graphs with five or fewer vertices which will yield a uniform type Lie algebra.

Our second class of Lie algebras, Heisenberg-like metric Lie algebras were defined by Gornet and Mast in their study of the spectral theory of nilmanifolds [10]. Heisenberg-like Lie algebras are a generalization of Heisenberg-Type metric Lie algebras which were defined by Kaplan [14]. Heisenberg-Type metric Lie algebras define highly symmetric nilmanifolds and solvmanifolds, which have been a rich source of examples for geometers [2]. This larger class also exhibits special geometric properties. In their study of Heisenberg-like Lie algebras, DeCoste, Meyer, and Mast characterized Heisenberg-like Lie groups in terms of curvature transformation and provided examples of Heisenberg-like Lie algebras which are not also Heisenberg-Type [5]. At this time, there are fewer than ten published examples of Heisenberg-like Lie algebras which are not also Heisenberg-Type [5, 10]. Due to the geometric definition of this class of Lie algebras, we study these Lie algebras over $\mathbb{R}$. In our study, we seek to construct new examples given in terms of a graph construction similar to that defined by uniform type Lie algebras.

Chapter 2

Background and Definitions

2.1 Lie Algebras

2.1.1 Lie Algebra Definitions and Basic Properties

Definition 2.1.1. Let $\mathbb{F}$ be a field. A *Lie algebra* $\mathfrak{g}$ is a vector space over the field $\mathbb{F}$ with an $\mathbb{F}$-bilinear map, $[\cdot,\cdot] : \mathfrak{g} \times \mathfrak{g} \to \mathfrak{g}$, called the *Lie bracket*, satisfying the following properties:

1. $[x, x] = 0$ for all $x \in \mathfrak{g}$

2. $[x, [y, z]] + [y, [z, x]] + [z, [x, y]] = 0$ for all $x, y, z \in \mathfrak{g}$ (*Jacobi identity*)

Remark. Let $\mathfrak{g}$ be a Lie algebra over a field $\mathbb{F}$. Then $[x, y] = -[y, x]$ for all $x, y \in \mathfrak{g}$ or equivalently, $[\cdot,\cdot] : \mathfrak{g} \times \mathfrak{g} \to \mathfrak{g}$ is an anti-commutative bilinear map.

Remark. Let $\mathfrak{g}$ be a vector space over a field $\mathbb{F}$ where $\mathrm{char}(\mathbb{F}) \neq 2$. Let $[\cdot,\cdot] : \mathfrak{g} \times \mathfrak{g} \to \mathfrak{g}$ be an $\mathbb{F}$-bilinear map. Then $\mathfrak{g}$ is a Lie algebra if and only if $[\cdot,\cdot] : \mathfrak{g} \times \mathfrak{g} \to \mathfrak{g}$ is an anti-commutative bilinear map which satisfies the Jacobi identity.

As a matter of convention, throughout the remainder of this thesis, we fix the field $\mathbb{F}$ and assume each Lie algebra is over the field $\mathbb{F}$.

Definition 2.1.2. Let $\mathfrak{g}$ be a Lie algebra with a vector subspace $\mathfrak{h}$. Then $\mathfrak{h}$ is a *subalgebra* of a Lie algebra $\mathfrak{g}$ if for all $x, y \in \mathfrak{h}$, $[x, y] \in \mathfrak{h}$.

Definition 2.1.3. Let $\mathfrak{g}$ be a Lie algebra and let $\mathfrak{i}$ be a subalgebra of $\mathfrak{g}$. If $[x, y] \in \mathfrak{i}$ for all $x \in \mathfrak{i}$, $y \in \mathfrak{g}$, then $\mathfrak{i}$ is called an *ideal* of $\mathfrak{g}$.

Definition 2.1.4. Let $\mathfrak{g}$ be a Lie algebra. Given a nonempty subset X of $\mathfrak{g}$ the *centralizer*, $C_{\mathfrak{g}}(X)$, is the set $C_{\mathfrak{g}}(X) = \{y \in \mathfrak{g} : [x, y] = 0 \text{ for all } x \in X\}$.

Lemma 2.1.5. Let $\mathfrak{g}$ be a Lie algebra, and let X be a nonempty subset of $\mathfrak{g}$. Then $C_{\mathfrak{g}}(X)$ is a subalgebra of $\mathfrak{g}$.

Definition 2.1.6. Let $\mathfrak{g}$ be a Lie algebra. The *center of* $\mathfrak{g}$ is the set $Z(\mathfrak{g}) = \{y \in \mathfrak{g} : [x, y] = 0 \text{ for all } x \in \mathfrak{g}\}$ or equivalently, $Z(\mathfrak{g}) = C_{\mathfrak{g}}(\mathfrak{g})$. Furthermore, we call a Lie algebra *abelian* when $Z(\mathfrak{g}) = \mathfrak{g}$.

Example 2.1.7. Let $\mathfrak{g}$ be a Lie algebra. The center $Z(\mathfrak{g})$ is an ideal of $\mathfrak{g}$.

Definition 2.1.8. A *homomorphism* from $\mathfrak{g}$ to $\mathfrak{h}$ is a linear map $\varphi : \mathfrak{g} \to \mathfrak{h}$ with the property $\varphi([x, y]) = [\varphi(x), \varphi(y)]$ for all $x, y \in \mathfrak{g}$. If φ is a bijection, then we say φ is an *isomorphism* and that $\mathfrak{g}$ and $\mathfrak{h}$ are *isomorphic*, and we write $\mathfrak{g} \cong \mathfrak{h}$.

Lemma 2.1.9. Let $\mathfrak{g}_1$ and $\mathfrak{g}_2$ be Lie algebras. Then let $\mathfrak{g} = \mathfrak{g}_1 \oplus \mathfrak{g}_2$ be the direct sum of their underlying vector spaces. If we define the $\mathbb{F}$-bilinear map,

$$[(x_1, x_2), (y_1, y_2)] = ([x_1, y_1], [x_2, y_2])$$

then $\mathfrak{g}$ becomes a Lie algebra.

Definition 2.1.10. Let $\mathfrak{g}_1$ and $\mathfrak{g}_2$ be Lie algebras over the field $\mathbb{F}$. Then we say $\mathfrak{g} = \mathfrak{g}_1 \oplus \mathfrak{g}_2$ with the bilinear operation defined in Lemma 2.1.9 is the *direct sum of* $\mathfrak{g}_1$ *and* $\mathfrak{g}_2$.

Lemma 2.1.11. Let $\mathfrak{g}_1$ and $\mathfrak{g}_2$ be Lie algebras and let $\mathfrak{g} = \mathfrak{g}_1 \oplus \mathfrak{g}_2$ be their direct sum. Then the following properties hold:

- $Z(\mathfrak{g}) = Z(\mathfrak{g}_1) \oplus Z(\mathfrak{g}_2)$
- $\dim(\mathfrak{g}) = \dim(\mathfrak{g}_1) + \dim(\mathfrak{g}_2)$ (as vector spaces)
- Let $\mathcal{B}_1$ be a basis of $\mathfrak{g}_1$ and let $\mathcal{B}_2$ be a basis of $\mathfrak{g}_2$. Then $\mathfrak{g}$ has a basis $\mathcal{B} = \mathcal{B}_1 \times \{0\} \cup \{0\} \times \mathcal{B}_2$.

Remark. Let $\mathfrak{g}_1$ and $\mathfrak{g}_2$ be Lie algebras. As a matter of convention we will use the direct sum to refer to both the external direct sum as $\mathfrak{g}_1 \oplus \mathfrak{g}_2 = \{(x, y) \mid x \in \mathfrak{g}_1, y \in \mathfrak{g}_2\}$ and internal direct sum as $\mathfrak{g}_1 \oplus \mathfrak{g}_2 = \{x + y \mid x \in \mathfrak{g}_1, y \in \mathfrak{g}_2\}$. It will be clear from context which is the intended use.

2.1.2 Nilpotent Lie Algebras

Definition 2.1.12. Let $\mathfrak{g}$ be a Lie algebra. Let $\mathfrak{i}$ and $\mathfrak{j}$ be ideals of $\mathfrak{g}$. We define the *product of ideals* as

$$[\mathfrak{i}, \mathfrak{j}] = \operatorname{span}\{[x, y] \mid x \in \mathfrak{i}, y \in \mathfrak{j}\}$$

In particular, $\mathfrak{g}$ is an ideal of $\mathfrak{g}$ and we call the product $\mathfrak{g}' = [\mathfrak{g}, \mathfrak{g}]$ the *commutator subalgebra* of $\mathfrak{g}$.

Lemma 2.1.13. Let $\mathfrak{g}$ be a Lie algebra. Let $\mathfrak{i}$ and $\mathfrak{j}$ be ideals of $\mathfrak{g}$. Then the product $[\mathfrak{i}, \mathfrak{j}]$ is an ideal of $\mathfrak{g}$.

Definition 2.1.14. We define the *lower central series* of a Lie algebra $\mathfrak{g}$ to be the series with terms

$$\mathfrak{g}^1 = \mathfrak{g}' \qquad \text{and} \qquad \mathfrak{g}^k = [\mathfrak{g}, \mathfrak{g}^{k-1}] \text{ for } k > 1$$

Definition 2.1.15. A Lie algebra $\mathfrak{g}$ is called *nilpotent* if $\mathfrak{g}^n = 0$ for some $n \in \mathbb{N}$. Furthermore, if $\mathfrak{g}$ is nilpotent and $n \in \mathbb{N}$ is such that $\mathfrak{g}^n = 0$ and $\mathfrak{g}^{n-1} \neq 0$, then we say $\mathfrak{g}$ is an *n-step nilpotent Lie algebra.*

Definition 2.1.16. Let $\mathbb{F} = \mathbb{C}$. Let $\mathfrak{g} = \mathfrak{v} \oplus \mathfrak{z}$ be a two-step nilpotent Lie algebra where $\mathfrak{z} = Z(\mathfrak{g})$. Furthermore, let $\langle \cdot, \cdot \rangle : \mathfrak{g} \times \mathfrak{g} \to \mathbb{R}$ be the standard inner product on $\mathbb{C}$. For each nonzero $z \in \mathfrak{z}$ define a skew-symmetric linear transformation $J_z : \mathfrak{v} \to \mathfrak{v}$ by

$$\langle [x, y], z \rangle = \langle J_z x, y \rangle \text{ for all } x, y \in \mathfrak{v}.$$

We call this map the *Kaplan J_z map.*

Lemma 2.1.17. Let $\mathfrak{g}$ be a two-step nilpotent Lie algebra. If $x, y, z \in \mathfrak{g}$, then $[x, [y, z]] = 0$. In particular, every term in the Jacobi identity of a two-step nilpotent Lie algebra is trivial.

To define a two-step nilpotent Lie algebra on a vector space $\mathfrak{g}$ with basis $\mathcal{B} = V \cup Z$, it suffices to define a Lie bracket $[\cdot, \cdot]$ on basis elements $v_i, v_j \in V$ by defining $[v_i, v_j] \in \mathrm{span}(Z)$ for all $i < j$, and assume further $[v_j, v_i] = -[v_i, v_j]$. Finally, define $[v_i, z_j] = [z_j, v_i] = [v_i, v_i] = 0$ for all remaining $v_i \in V$ and for all $z_j \in Z$. Then by extending $[\cdot, \cdot]$ bilinearly we define a Lie bracket on $\mathfrak{g}$.

We simplify our discussion of two-step nilpotent Lie algebras by defining only the non-zero Lie brackets $[v_i, v_j] \in Z$ and assume $[v_j, v_i] = -[v_i, v_j]$. It will then be assumed all other undefined Lie brackets on basis elements are trivial. We use this simplification henceforth.

Example 2.1.18. Let $\mathcal{B} = \{v_1, v_2, v_3\} \cup \{z_1, z_2, z_3\}$ and let $\mathfrak{f}_{3,2}$ be the span of this basis. Define a Lie bracket on $\mathfrak{f}_{3,2}$ by

$$[v_1, v_2] = z_1 \qquad [v_2, v_3] = z_2 \qquad [v_3, v_1] = z_3.$$

Then $\mathfrak{f}_{3,2}$ is a Lie algebra. We call this Lie algebra the *free two-step nilpotent Lie algebra on three generators.*

Example 2.1.19. Generalizing Example 2.1.18, let $\mathcal{B} = \{v_1, ..., v_n\} \cup \{z_1, ..., z_{\binom{n}{2}}\}$. Let $\mathfrak{f}_{n,2}$ be the span of this basis and set

$$[v_i, v_j] = z_{\{i,j\}}$$

where the vectors $z_1, ..., z_{\binom{n}{2}}$ have been relabeled $z_{\{i,j\}}$ for all possible two element subsets $\{i, j\}$ of $\{1, ..., n\}$. We call this Lie algebra the *free two-step nilpotent Lie algebra on n generators*.

2.2 Graph Theory

2.2.1 Graph Definitions and Basic Properties

Definition 2.2.1. An *undirected graph* is an ordered pair $G = (V, E)$ comprised of a non-empty set of vertices V and a set of edges $E \subset \{\{v_i, v_j\} \mid v_i, v_j \in V\}$. A *directed graph* is an ordered pair $G = (V, E)$ comprised of a non-empty set of vertices V and a set of edges $E \subset \{(v_i, v_j) \mid v_i, v_j \in V\}$.

Definition 2.2.2. Let $G = (V, E)$ be an undirected graph. Two vertices, v_i and v_j, are *adjacent* if there exists an edge $\{v_i, v_j\} \in E$. We say two edges $\{v_i, v_j\}$ and $\{v_k, v_l\}$ are *adjacent* if

$$\{v_i, v_j\} \neq \{v_k, v_l\} \text{ and } \{v_i, v_j\} \cap \{v_k, v_l\} \neq \varnothing.$$

An edge $\{v_i, v_j\}$ and a vertex v_k are *incident* if $v_k \in \{v_i, v_j\}$.

In a similar way, we naturally extend these definitions for a directed graph. In particular, if G is a directed graph, two vertices v_i and v_j are *adjacent* if there exists an edge (v_i, v_j) or $(v_j, v_i) \in E$. We say two edges (v_i, v_j) and (v_k, v_l) are *adjacent* if $\{v_i, v_j\} \neq \{v_k, v_l\}$ and $\{v_i, v_j\} \cap \{v_k, v_l\} \neq 0$. Finally, we say an edge (v_i, v_j) and a vertex v_k are *incident* if $v_k = v_i$ or $v_k = v_j$.

Let $G = (V, E)$ be an undirected (resp. directed) graph. We say the *degree of a vertex* $v \in V$ is the number of edges which are incident to the vertex v. An undirected (resp. directed) graph $G = (V, E)$ is *regular* if all vertices have the same degree; a regular undirected (resp. directed) graph of degree k is also called a k-regular undirected (resp. directed) graph.

Directed graphs and undirected graphs share many similar results. To simplify our discussion, we use the term *graph* to allow for cases in which a directed or an undirected graph may be used.

A graph is *finite* if $|V|$ and $|E|$ are finite. We say a graph is *trivial* if the graph has exactly one vertex and no edges. Finally, we say a graph is *simple* if there are no self adjacent vertices. As a matter of convention, it will be assumed that every graph will be finite and non-trivial.

Definition 2.2.3. Let $G = (V, E)$ be a graph. We say a graph $H = (V', E')$ is a *subgraph of* G if $V' \subset V$ and $E' \subset E$.

Definition 2.2.4. Let $G_1 = (V_1, E_1)$ and $G_2 = (V_2, E_2)$ be undirected graphs. A *graph isomorphism* from G_1 to G_2 is a bijection $\varphi : V_1 \to V_2$ with the property $\{v_i, v_j\} \in E_1$ if and only if $\{\varphi(v_1), \varphi(v_2)\} \in E_2$. Let $G_1 = (V_1, E_1)$ and $G_2 = (V_2, E_2)$ be directed graphs. A *graph isomorphism* from G_1 to G_2 is a bijection $\varphi : V_1 \to V_2$ with the property $(v_i, v_j) \in E_1$ if and only if $(\varphi(v_i), \varphi(v_j)) \in E_2$. If there exists a graph isomorphism φ from G_1 to G_2 we say G_1 and G_2 are *isomorphic* and write $G_1 \simeq G_2$.

Definition 2.2.5. Let $G = (V, E)$ be a graph. An *edge coloring* is a surjective function $c : E \to Z$ such that no two adjacent edges share the same color. Furthermore, we say the *number of colors* is $|Z|$.

Definition 2.2.6. Let $G_1 = (V_1, E_1)$ and $G_2 = (V_2, E_2)$ be undirected graphs with edge colorings $c_1 : E_1 \to Z_1$ and $c_2 : E_2 \to Z_2$ respectively. Let φ be an isomorphism from G_1

to G_2. We say G_1 and G_2 are *isomorphic as edge colored graphs* if there exists a bijection $\theta : Z_1 \to Z_2$ with the property that all $\{v_i, v_j\} \in E_1$, $\theta(c_1(\{v_i, v_j\})) = c_2(\{\varphi(v_i), \varphi(v_j)\})$.

Let $G_1 = (V_1, E_1)$ and $G_2 = (V_2, E_2)$ be directed graphs with edge colorings $c_1 : E_1 \to Z_1$ and $c_2 : E_2 \to Z_2$ respectively. Let φ be an isomorphism from G_1 to G_2. We say G_1 and G_2 are *isomorphic as edge colored graphs* if there exists a bijection $\theta : Z_1 \to Z_2$ with the property that for all $(v_i, v_j) \in E_1$, $\theta(c_1(v_i, v_j)) = c_2(\varphi(v_i), \varphi(v_j))$.

Definition 2.2.7. Let $G_1 = (V_1, E_1)$ and $G_2 = (V_2, E_2)$ be graphs. We say G_1 and G_2 are *disjoint* if $V_1 \cap V_2 = \varnothing$.

Definition 2.2.8. Let $G_1 = (V_1, E_1)$ and $G_2 = (V_2, E_2)$ be graphs. The *graph union* of G_1 and G_2 is the graph $G = (V_1 \cup V_2, E_1 \cup E_2)$.

Definition 2.2.9. Let $G = (V, E)$ be a graph and let $n = |V|$ be the number of vertices. Define an $n \times n$ matrix A by,

$$(A)_{i,j} = \begin{cases} 1 & \text{if } v_i \text{ is adjacent to } v_j \\ 0 & \text{otherwise} \end{cases}$$

We call this matrix the *adjacency matrix of the graph G*.

Chapter 3

Uniform Lie Algebras and Uniform Graphs

3.1 Uniform Lie Algebras: Examples and Properties

3.1.1 Definitions and Examples

Uniform metric Lie algebras were first defined by DeLoff in his study of Einstein solvmanifolds in [6].

Definition 3.1.1. Let $\mathfrak{g}$ be a Lie algebra. Then $\mathfrak{g}$ is said to be a *uniform Lie algebra of type* (p, q, r) where $p, q, r > 0$, if there exists a basis

$$\mathcal{B} = \{v_1, \ldots, v_q\} \cup \{z_1, \ldots, z_p\}$$

of $\mathfrak{g}$ and an integer $s > 0$ such that

1. $[v_i, v_j] \in \{0, \pm z_1, \ldots, \pm z_p\}$ for all i, j.
2. If $[v_i, v_j] \neq 0$ and $[v_i, v_j] = \pm[v_i, v_k]$, then $j = k$.
3. For every z_l there are exactly r disjoint pairs $\{v_i, v_j\}$ with $[v_i, v_j] = \pm z_l$.
4. For every v_i there are exactly s vectors v_j such that $[v_i, v_j] \neq 0$.

The basis $\mathcal{B}$ of $\mathfrak{g}$ is called a *uniform basis*. If we let $V = \{v_1, \ldots, v_q\}$ and $Z = \{z_1, \ldots, z_p\}$, and $\mathfrak{v} = \mathrm{span}(V)$ and $\mathfrak{z} = \mathrm{span}(Z)$, then $\mathfrak{g} = \mathfrak{v} \oplus \mathfrak{z}$.

Remark. If $\mathfrak{g}$ is a uniform Lie algebra with uniform basis $\mathcal{B}$ endowed with a non-degenerate (symmetric) $\mathbb{F}$-bilinear form $\langle \cdot, \cdot \rangle$ making $\mathcal{B}$ orthonormal, then $\mathfrak{g}$ is a *uniform metric Lie algebra.* Furthermore, it will be shown that all uniform Lie algebras are two-step nilpotent and thus satisfy the Jacobi identity trivially.

Before we begin our discussion of uniform Lie algebras, we introduce some examples. In our examples, it suffices to define nonzero $[v_i, v_j]$ with $i < j$ then obtain $[v_j, v_i] = -[v_i, v_j]$ by anti-commutativity. It is assumed that all other Lie brackets are trivial. We then extend bilinearly and obtain a Lie algebra.

Example 3.1.2. Let $\mathfrak{h}_{2q+1}$ be the span of the basis $\mathcal{B} = \{v_1, \ldots, v_{2q}\} \cup \{z_1\}$ and define a Lie bracket on $\mathfrak{h}_{2q+1}$ by $[v_1, v_2] = [v_3, v_4] = \cdots = [v_{2q-1}, v_{2q}] = z_1$. Then $\mathfrak{h}_{2q+1}$ is a uniform Lie algebra of type $(1, 2q, q)$. We call this Lie algebra the *Heisenberg algebra* of dimension $2q + 1$.

The following example demonstrates that a uniform Lie algebra does not necessarily have a unique uniform basis.

Example 3.1.3. Let $\mathfrak{h}_3$ be the three dimensional Heisenberg Lie algebra. By Example 3.1.2, the basis $\mathcal{B} = \{v_1, v_2\} \cup \{z_1\}$ is a uniform basis of type $(1, 2, 1)$ and Lie bracket defined by $[v_1, v_2] = z_1$.

However, if we define the basis $\mathcal{B}' = \{v_1 + v_2, v_1 - v_2\} \cup \{-2z_1\}$ of $\mathfrak{h}_3$, then $\mathcal{B}'$ is also a uniform basis of type $(1, 2, 1)$ with $[v_1 + v_2, v_1 - v_2] = -2z_1$.

Example 3.1.4. Let $\mathfrak{g}$ the span of the basis $\mathcal{B} = \{v_1, v_2, v_3, v_4\} \cup \{z_1, z_2\}$ and define a Lie bracket on $\mathfrak{g}$ by $[v_1, v_2] = z_1$ and $[v_3, v_4] = z_2$. Then $\mathfrak{g}$ is a uniform Lie algebra of type $(2, 4, 1)$. Furthermore, $\mathfrak{g} \cong \mathfrak{h}_3 \oplus \mathfrak{h}_3$ where $\mathfrak{h}_3$ is the three dimensional Heisenberg algebra defined in Example 3.1.2.

Example 3.1.5. Let $\mathfrak{h}$ be the span of the basis $\mathcal{B} = \{v_1, v_2, v_3, v_4\} \cup \{z_1, z_2\}$ and define a Lie bracket on $\mathfrak{g}$ by $[v_1, v_2] = [v_3, v_4] = z_1$ and $[v_1, v_3] = [v_2, v_4] = z_2$. Then $\mathfrak{h}$ is a uniform Lie algebra of type $(2, 4, 2)$.

Example 3.1.6. The free two-step nilpotent Lie algebra on n generators from Example 2.1.19 is a uniform Lie algebra of type $(\binom{n}{2}, n, 1)$.

3.1.2 Basic Properties of Uniform Lie Algebras

Lemma 3.1.7. Let $\mathfrak{g}$ be a uniform Lie algebra of type (p, q, r). If $v_i \in V$ then the set $\{[v_i, v_j] \mid v_j \in V \text{ and } [v_i, v_j] \neq 0\}$ is a linearly independent subset of $\{\pm z_1, ..., \pm z_p\}$. Furthermore $| \{[v_i, v_j] \mid v_j \in V \text{ and } [v_i, v_j] \neq 0\}| = s$.

Proof. Let $v_i \in V$. Let $A_i = \{[v_i, v_j] \mid v_j \in V \text{ and } [v_i, v_j] \neq 0\}$. In the definition of a uniform Lie algebra, the number s is assumed to be positive; then there exists at least one v_j such that $[v_i, v_j] \neq 0$ and therefore $A_i \neq \varnothing$.

Let $[v_i, v_j] \in A_i$. Then $[v_i, v_j] \neq 0$. By definition of a uniform Lie algebra $[v_i, v_j] \in \{0, \pm z_1, ..., \pm z_p\}$; thus $A_i \subset \{\pm z_1, ..., \pm z_p\}$. Assume $z_l, -z_l \in A_i$. Then there exists $[v_i, v_j] = z_l$ and $[v_i, v_k] = -z_l$. By part 2 of the definition of a uniform Lie algebra, since $[v_i, v_j] = -[v_i, v_k]$ it follows $v_j = v_k$. A contradiction arises as $v_j = v_k$, $[v_i, v_j] = z_l$ and $[v_i, v_k] = -z_l$, but $z_l \neq 0$. Thus if $z_l \in A_i$, $-z_l \notin A_i$ and if $-z_l \in A_i$, $z_l \notin A_i$. Therefore, by linear independence of $\{z_1, ..., z_p\}$, A_i is linearly independent.

Finally, by definition of a uniform Lie algebra, for v_i there exist exactly s vectors $v_j \in V$ such that $[v_i, v_j] \neq 0$. It follows $|A_i| = s$. $\square$

Theorem 3.1.8. Let $\mathfrak{g}$ be a uniform Lie algebra of type (p, q, r). Then $Z(\mathfrak{g}) = \mathfrak{z}$.

Proof. First, we show $Z(\mathfrak{g}) \subset \mathfrak{z}$; then we show $\mathfrak{z} \subset Z(\mathfrak{g})$.

Let $x \in Z(\mathfrak{g})$. Write $x = \sum_{i=1}^{q} \alpha_i v_i + \sum_{i=1}^{p} \beta_i z_i$. Assume there exists $1 \leq i \leq q$ such that $\alpha_i \neq 0$. By definition of a uniform Lie algebra there exists $v_j \in V$ such that

$[v_i, v_j] \neq 0$. Then

$$\begin{aligned}
[x, v_j] &= \left[\sum_{i=1}^{q} \alpha_i v_i + \sum_{i=1}^{p} \beta_i z_i, v_j\right] \\
&= \sum_{i=1}^{q} \alpha_i [v_i, v_j] + \sum_{i=1}^{p} \beta_i [z_i, v_j] \\
&= \sum_{i=1}^{q} \alpha_i [v_i, v_j] \\
&= \sum_{i=1}^{q} -\alpha_i [v_j, v_i].
\end{aligned}$$

Since $x \in Z(\mathfrak{g})$ it follows that $\sum_{i=1}^{q} -\alpha_i [v_j, v_i] = 0$. By Lemma 3.1.7 the set $\{[v_j, v_i] \mid v_i \in V \text{ and } [v_j, v_i] \neq 0\}$ is linearly independent. In particular, if $[v_j, v_i] \neq 0$, then $\alpha_i = 0$. A contradiction then arises as it was assumed $[v_j, v_i] \neq 0$ and $\alpha_i \neq 0$. Therefore $\alpha_i = 0$ for all $1 \leq i \leq q$. Hence $x \in \mathfrak{z}$ as desired.

Now, let $x = \sum_{k=1}^{p} \beta_k z_k \in \mathfrak{z}$. Let $y \in \mathfrak{g}$. Since $\mathfrak{g} = \mathfrak{v} \oplus \mathfrak{z}$, write $y = \sum_{i=1}^{q} \alpha_i' v_i + \sum_{l=1}^{p} \beta_l' z_l$. Then

$$\begin{aligned}
[x, y] &= \left[\sum_{k=1}^{p} \beta_k z_k,\ \sum_{i=1}^{q} \alpha_i' v_i + \sum_{l=1}^{p} \beta_l' z_l\right] \\
&= \left[\sum_{k=1}^{p} \beta_k z_k,\ \sum_{i=1}^{q} \alpha_i' v_i\right] + \left[\sum_{k=1}^{p} \beta_k z_k, \sum_{l=1}^{p} \beta_l' z_l\right] \\
&= \sum_{k=1}^{p} \sum_{i=1}^{q} [\beta_k z_k, \alpha_i' v_i] + \sum_{k=1}^{p} \sum_{l=1}^{p} [\beta_k z_k, \beta_l' z_l] \\
&= \sum_{k}^{p} \sum_{i=1}^{q} \beta_k \alpha_i' [z_k, v_i] + \sum_{k=1}^{p} \sum_{l=1}^{p} \beta_k \beta_l' [z_k, z_l] \\
&= 0.
\end{aligned}$$

Since $[x, y] = 0$ for all $y \in \mathfrak{g}$, $x \in Z(\mathfrak{g})$. Thus we have shown $\mathfrak{z} = \operatorname{span}(Z) = Z(\mathfrak{g})$. □

The next corollary shows that p and q are unique for a uniform Lie algebra.

Corollary 3.1.9. Let $\mathfrak{g}$ be a uniform Lie algebra of type (p, q, r) and of type (p', q', r'). Then $p = p'$ and $q = q'$.

Proof. Since $\dim(Z(\mathfrak{g})) = p$ and $\dim(Z(\mathfrak{g})) = p'$, $p = p'$. Then since $\dim(\mathfrak{g}) = p + q = p' + q'$, it follows that $q = q'$. $\square$

Lemma 3.1.10. Let $\mathfrak{g}$ be a uniform Lie algebra of type (p, q, r) with uniform basis $\mathcal{B} = V \cup Z$. Let $\mathfrak{h}$ be a Lie algebra. If $\mathfrak{g} \cong \mathfrak{h}$, then $\mathfrak{h}$ has uniform basis of type (p, q, r).

Proof. Let $\varphi : \mathfrak{g} \to \mathfrak{h}$ be an isomorphism. Then $\varphi(\mathcal{B}) = \varphi(V) \cup \varphi(Z)$ is a basis for $\mathfrak{h}$. We will show $\varphi(\mathcal{B})$ satisfies the defining criteria of a uniform basis for $\mathfrak{h}$.

1. Let $\varphi(v_i), \varphi(v_j) \in \varphi(V)$. Then $[\varphi(v_i), \varphi(v_j)] = \varphi([v_i, v_j])$ and $\varphi([v_i, v_j]) \in \varphi(\{0, \pm z_1, ..., \pm z_p\}) = \{0, \pm\varphi(z_1), ..., \pm\varphi(z_p)\}$.

2. Let $\varphi(v_i), \varphi(v_j) \in \varphi(V)$. If $[\varphi(v_i), \varphi(v_j)] \neq 0$ and $[\varphi(v_i), \varphi(v_j)] = \pm[\varphi(v_i), \varphi(v_k)]$ then $\varphi([v_i, v_j]) = \varphi(\pm[v_i, v_k])$. Then $[v_i, v_j] = \pm[v_i, v_k]$ and $v_j = v_k$. Thus $\varphi(v_j) = \varphi(v_k)$.

3. Let $\varphi(z_l) \in \varphi(Z)$. By definition there exist exactly r disjoint pairs $\{v_i, v_j\}$ such that $[v_i, v_j] = \pm z_l$. Then since φ is an isomorphism, there are exactly r disjoint pairs $\{\varphi(v_i), \varphi(v_j)\}$ such that $[\varphi(v_i), \varphi(v_j)] = \varphi([v_i, v_j]) = \pm\varphi(z_l)$.

4. Let $\varphi(v_i) \in \varphi(V)$. By definition there exist s vectors v_j such that $[v_i, v_j] \neq 0$. Then since φ is an isomorphism, $\ker(\varphi) = 0$ and in particular $\varphi([v_i, v_j]) = 0$ if and only if $[v_i, v_j] = 0$. Thus there exist s vectors $\varphi(v_j)$ such that $[\varphi(v_i), \varphi(v_j)] \neq 0$.

Thus we have proven that $\mathfrak{h}$ is a uniform Lie algebra of type (p, q, r) with uniform basis $\varphi(\mathcal{B}) = \{\varphi(v_1), ..., \varphi(v_q)\} \cup \{\varphi(z_1), ..., \varphi(z_p)\}$. $\square$

Remark. The statement of Lemma 3.1.10 does not prove in general that there is a unique type (p, q, r) for every uniform Lie algebra. In particular, it remains unknown if a uniform Lie algebra can have two uniform bases $\mathcal{B}_1$ and $\mathcal{B}_2$ of type (p, q, r_1) and (p, q, r_2) respectively, where $r_1 \neq r_2$.

Lemma 3.1.11. Let $\mathfrak{g}$ be a uniform Lie algebra of type (p, q, r). Then the commutator algebra $\mathfrak{g}' = \mathfrak{z}$ and $\mathfrak{g}$ is a two-step nilpotent Lie algebra. In particular $\mathfrak{g}' \neq 0$ and $\mathfrak{g}^2 = 0$.

Proof. By part 1 of the definition of a uniform Lie algebra, $[v_i, v_j] \in \mathfrak{z}$ for all $v_i, v_j \in V$. Let $z_l \in Z$. Then by part 3 of the definition of a uniform Lie algebra, there exists at least one pair of vectors $\{v_i, v_j\}$ such that $[v_i, v_j] = \pm z_l$. Thus $z_l \in \mathfrak{g}'$. Thus it follows that $\mathfrak{z} = \text{span}\,(Z) = \mathfrak{g}'$.

Then $\mathfrak{g}^2 = [\mathfrak{g}, \mathfrak{g}'] = [\mathfrak{g}, \mathfrak{z}] = \{0\}$. Thus $\mathfrak{g}' \neq 0$, $\mathfrak{g}^2 = 0$, and $\mathfrak{g}$ is a 2-step nilpotent Lie algebra. $\square$

Remark. Assume $\mathbb{F} = \mathbb{R}$. Let $\mathfrak{g}$ be a uniform metric Lie algebra of type (p, q, r) with basis $\mathcal{B} = V \cup Z$ and inner product $\langle \cdot, \cdot \rangle$ which makes $\mathcal{B}$ orthonormal. Let $z_l \in Z$. Let J_{z_l} be the skew-symmetric linear transformation defined in Definition 2.1.16. Let $v_i, v_j \in V$. Then $\langle [v_i, v_j], z_l \rangle = \langle J_{z_l} v_i, v_j \rangle$. In particular,

$$J_{z_l}(v_i) = \sum_{j=1}^{q} \varepsilon_j v_j \qquad \text{where} \qquad \varepsilon_j = \begin{cases} 1 & \text{if } [v_i, v_j] = z_l \\ -1 & \text{if } [v_i, v_j] = -z_l \\ 0 & \text{otherwise} \end{cases} \tag{3.1}$$

for all $v_i \in V$. By part 3 of the definition of a uniform Lie algebra, there exist exactly r disjoint pairs $\{v_k, v_l\}$ such that $[v_k, v_l] = \pm z_l$. It follows $J_{z_l}(v_i) = \varepsilon_j v_j$.

Lemma 3.1.12. Assume $\mathbb{F} = \mathbb{R}$. Let $\mathfrak{g}$ be a uniform Lie algebra of type (p, q, r) and let $z_l \in Z$. Let J_{z_l} be the skew-symmetric linear transformation defined in Definition 2.1.16. Then $rk(J_{z_l}) = 2r$.

Proof. By the previous remark, for all v_i, v_j, $J_{z_l}(v_i) = \varepsilon_j v_j$ where ε_j is defined in Equation 3.1. Therefore it follows that the image $Im(J_{z_l}) = \text{span}\{v_j \mid [v_i, v_j] = \pm z_l \text{ for some } v_i \in V\}$. By part 3 of the definition of uniform Lie algebras, there are exactly r disjoint pairs $\{v_i, v_j\}$ such that $[v_i, v_j] = \pm z_l$ thus $rk(J_{z_l}) = \dim(Im(J_{z_l})) = 2r$. $\square$

Lemma 3.1.13. Let $\mathfrak{g}_1$ be a uniform Lie algebra of type (p_1, q_1, r_1) with uniform basis $\mathcal{B}_1$ and let $\mathfrak{g}_2$ be a uniform Lie algebras of type (p_2, q_2, r_2) with uniform basis $\mathcal{B}_2$. Let $\mathfrak{g} = \mathfrak{g}_1 \oplus \mathfrak{g}_2$ be the direct sum with basis $\mathcal{B} = \mathcal{B}_1 \cup \mathcal{B}_2$. Then $\mathcal{B}$ is a uniform basis of type (p, q, r) if and only if $r_1 = r_2$ and $s_1 = s_2$. Furthermore if $\mathfrak{g}$ a uniform Lie algebra, then $\mathfrak{g}$ is of type $(p_1 + p_2, q_1 + q_2, r)$ where $r = r_1 = r_2$.

Proof. As a matter of notation, let $\mathcal{B}_1 = V_1 \cup Z_1$ and let $\mathcal{B}_2 = V_2 \cup Z_2$. Then $\mathcal{B} = (V_1 \cup V_2) \cup (Z_1 \cup Z_2)$. Since $\mathfrak{g}$ is the direct sum of $\mathfrak{g}_1$ and $\mathfrak{g}_2$, $\mathfrak{g}_1 \bigcup \mathfrak{g}_2 = \{0\}$. In particular, $V_1 \bigcap V_2 = \varnothing$ and $Z_1 \bigcap Z_2 = \varnothing$.

Assume $\mathcal{B}$ is a uniform basis of type (p, q, r). Let $z_l \in Z = Z_1 \cup Z_2$. Then $z_l \in Z_1$ or $z_l \in Z_2$. Assume $z_l \in Z_1$. Since $\mathcal{B}_1$ is a uniform basis, there exist exactly r_1 disjoint pairs $\{v_i, v_j\} \subset V_1$ such that $[v_i, v_j] = \pm z_l$. Furthermore, since $\mathfrak{g}$ is a direct sum, there are no pairs $v_i, v_j \in V_2$ such that $[v_i, v_j] = \pm z_l \in Z_1$. Thus, there are exactly $r = r_1$ disjoint pairs $\{v_i, v_j\} \subset V_1 \subset V$ such that $[v_i, v_j] = \pm z_l$. Similarly, if $z_l \in Z_2$ then $r_2 = r$. Therefore $r_1 = r = r_2$.

Let $v_i \in V = V_1 \cup V_2$. Then $v_i \in V_1$ or $v_i \in V_2$. Assume $v_i \in V_1$. Since $\mathcal{B}_1$ is a uniform basis, there are exactly s_1 vectors $v_j \in V_1$ such that $[v_i, v_j] \neq 0$. Because $\mathfrak{g}$ is a direct sum, $[v_i, v_j] = 0$ for all $v_j \in V_2$. We conclude there are exactly s_1 vectors $v_j \in V$ such that $[v_i, v_j] \neq 0$ and $s_1 = s$. Similarly, if $v_i \in V_2$ it follows $s_2 = s$. Therefore $s = s_1 = s_2$.

Thus if $\mathfrak{g}$ is a uniform Lie algebra, it has uniform basis $\mathcal{B} = (V_1 \cup V_2) \cup (Z_1 \cup Z_2)$ of type $(p_1 + p_2, q_1 + q_2, r)$ where $r = r_1 = r_2$.

Assume $r_1 = r_2$ and $s_1 = s_2$. Then we show $\mathfrak{g}$ is a uniform Lie algebra.

1. Let $v_i, v_j \in V = V_1 \cup V_2$. Rewrite $Z_i = \{z_1^i, ..., z_{p_i}^i\}$ for $i = 1, 2$. If $v_i, v_j \in V_1$ then by

definition of a uniform Lie algebra, $[v_i, v_j] \in \{0, \pm z_1^1, ..., \pm z_{p1}^1\}$. Similarly if $v_i, v_j \in V_2$ then by definition of a uniform Lie algebra of $\mathfrak{g}_2$, $[v_i, v_j] \in \{0, \pm z_1^2, ..., \pm z_{p_2}^2\}$. Since $\mathfrak{g}$ is a direct sum, if $v_i \in V_1$ and $v_j \in V_2$ or if $v_i \in V_2$ and $v_j \in V_1$, then $[v_i, v_j] = 0$. Therefore $[v_i, v_j] \in \{0, \pm z_1^1, ..., \pm z_{p_1}^1, \pm z_1^2, ..., \pm z_{p_2}^2\}$ for all $v_i, v_j \in V$.

2. Let $[v_i, v_j] \neq 0$ for $v_i, v_j \in V = V_1 \cup V_2$. Assume further let $[v_i, v_j] = \pm[v_i, v_k]$ for some $v_k \in V$. By definition of a direct sum, if $[v_i, v_j] \neq 0$ then $v_i, v_j \in V_1$ or $v_i, v_j \in V_2$. Without loss of generality assume $v_i, v_j \in V_1$. Since $[v_i, v_k] \neq 0$, $v_k \in V_1$. By definition, since $\mathfrak{g}_1$ is a uniform Lie algebra, if $[v_i, v_j] = \pm[v_i, v_k]$ it follows $v_j = v_k$. Thus if $[v_i, v_j] = \pm[v_i, v_k]$ for some $v_k \in V$, $v_j = v_k$.

3. Let $z_l \in Z = Z_1 \cup Z_2$. If $z_l \in Z_1$ there are exactly r_1 disjoint pairs $\{v_i, v_j\} \subset V_1$ such that $[v_i, v_j] = \pm z_l$. If $z_l \in Z_2$ there are exactly r_2 disjoint pairs $\{v_i, v_j\} \subset V_2$ such that $[v_i, v_j] = \pm z_l$. Since $\mathfrak{g} = \mathfrak{g}_1 \oplus \mathfrak{g}_2$ is a direct sum, there are no pairs $\{v_i, v_j\}$ where $v_i \in V_1$ and $v_j \in V_2$ such that $[v_i, v_j] = \pm z_l$. Thus it follows for each $z_l \in Z$, there are exactly $r = r_1 = r_2$ disjoint pairs $\{v_i, v_j\}$ such that $[v_i, v_j] = \pm z_l$.

4. Let $v_i \in V = V_1 \cup V_2$. If $v_i \in V_1$ there are exactly s_1 vectors $v_j \in V_1$ such that $[v_i, v_j] \neq 0$. If $v_i \in V_2$ there are exactly s_2 vectors v_j such that $[v_i, v_j] \neq 0$. By definition of a direct sum if $v_i \in V_1$ and $v_j \in V_2$ or $v_i \in V_2$ and $v_j \in V_1$, then $[v_i, v_j] = 0$. Thus for every $v_i \in V = V_1 \cup V_2$ there are exactly $s = s_1 = s_2$ vectors v_j such that $[v_i, v_j] \neq 0$.

Therefore $\mathcal{B}$ is a uniform basis. □

3.1.3 Relations Between p, q, r, and s

Lemma 3.1.14. *Let $\mathfrak{g}$ be a uniform Lie algebra of type (p, q, r), Let $A = \{(i, j) \mid [v_i, v_j] \neq 0\} \subset \mathbb{N} \times \mathbb{N}$, and let s be as defined in the definition of uniform Lie algebra. Then the numbers $p, q, r,$ and s are related by the equation $sq = 2rp$.*

Proof. Let $|A|$ denote the cardinality of A. We will first show that $sq = |A|$, then show $2rp = |A|$, and thus prove that $2rp = sq$.

For all v_i, there exist exactly s basis vectors v_j such that, $[v_i, v_j] \neq 0$. Thus the value of $|A|$ is sq, as there are q elements in V.

All nontrivial $[v_i, v_j]$ are of the form $\pm z_l$. For all z_l there exist r disjoint pairs $\{v_i, v_j\} \subseteq V$ such that $[v_i, v_j] = \pm z_l$. Since these pairs are disjoint and there are exactly r of these pairs, it follows for each z_l there are $2r$ possible Lie brackets such that $[v_i, v_j] = -[v_j, v_i]$ where $[v_i, v_j] = \pm z_l$. Thus there are $2rp$ as there are p elements in Z and each element has exactly $2r$ brackets. Thus $2rp = |\{(v_i, v_j) \mid [v_i, v_j] \neq 0\}| = |A|$.

Therefore $sq = 2rp = |A|$. $\square$

Lemma 3.1.15. Let $\mathfrak{g}$ be a uniform Lie algebra of type (p, q, r). Then $rp \leq \binom{q}{2}$.

Proof. Let $v_i \in V$. By definition of a uniform Lie algebra there exist exactly s vectors v_j such that $[v_i, v_j] \neq 0$. Thus $s \leq q - 1$. Then $\frac{sq}{2} \leq \frac{q(q-1)}{2} = \binom{q}{2}$. By Lemma 3.1.14, $2rp = sq$. In particular, $rp = \frac{sq}{2}$ and $rp \leq \binom{q}{2}$. $\square$

Corollary 3.1.16. Let $\mathfrak{g}$ be a uniform Lie algebra of type (p, q, r), then $s \leq p \leq \frac{sq}{2} = rp$.

Proof. First we show $p \leq rp$. Then we show $s \leq p$. By definition of a uniform Lie algebra, $r \geq 1$. Therefore by Lemma 3.1.14, $p \leq rp = \frac{sq}{2}$.

Let $v_i \in V$. By Lemma 3.1.7, $\{[v_i, v_j] \mid v_j \in V \text{ and } [v_i, v_j] \neq 0\} \subset \{\pm z_1, \ldots, \pm z_p\}$ is independent and $s =\mid \{[v_i, v_j] \mid v_j \in V \text{ and } [v_i, v_j] \neq 0\} \mid$. Therefore, $s \leq p$. $\square$

Lemma 3.1.17. If $\mathfrak{g}$ is a uniform Lie algebra of type (p, q, r), $\dim(\mathfrak{g}) \geq 3$. In particular, $q \geq 2$.

Proof. In the definition of a uniform Lie algebra $q \geq 1$ and $p \geq 1$. Since $|V| = q$ and $|Z| = p$, V and Z are nonempty. Let $v_i \in V$. By definition of a uniform Lie algebra, $s \geq 1$. Therefore, there exist at least one $v_j \in V$ such that $[v_i, v_j] \neq 0$ where $v_i \neq v_j$. Thus $\dim(\mathfrak{v}) = q \geq 2$. Since $\dim(\mathfrak{g}) = p + q$, $\dim(\mathfrak{g}) \geq 3$. $\square$

Lemma 3.1.18. Let $\mathfrak{g}$ be a uniform Lie algebra $(2, 4, r)$, then $r = 1$ or 2.

Proof. Proceed by contradiction, assume that $r \geq 3$. Let $z_l \in Z$. Since $\mathfrak{g}$ is a uniform Lie algebra and $r \geq 3$, there exist at least 3 disjoint pairs, $\{v_{i_k}, v_{j_k}\}$ such that $[v_{i_k}, v_{j_k}] = \pm z_l$ where $1 \leq k \leq r$; in particular, there exist 6 or more distinct elements of V. Thus we arrive at a contradiction as $|V| = q = 4$. Therefore $r = 1$ or $r = 2$. $\square$

Lemma 3.1.19. If $\mathfrak{g}$ is a uniform Lie algebra of type $(p, 2, r)$ then $p = r = 1$.

Proof. By definition $p \in \mathbb{N}$. Let $z_l \in Z$. Since $r \geq 1$, there exist at least one disjoint pair $\{v_i, v_j\}$ such that $[v_i, v_j] = \pm z_l$. By hypothesis $q = 2$, therefore $V = \operatorname{span}\{v_i, v_j\}$ and $s = 1$. By Lemma 3.1.14, $2rp = 2$. It follows $r = p = 1$. $\square$

Lemma 3.1.20. Let $\mathfrak{g}$ be a uniform Lie algebra of type (p, q, r). Then for any $v_i \in \mathfrak{v}$, the centralizer, $C_{\mathfrak{g}}(\{v_i\}) = \operatorname{span}(\{v_j \mid [v_i, v_j] = 0\}) + \mathfrak{z}$; furthermore $\{v_j \mid [v_i, v_j] = 0\} \cup Z$ is a basis of $C_{\mathfrak{g}}(\{v_i\})$.

Proof. We proceed by showing two way containment; first, showing $C_{\mathfrak{g}}(\{v_i\}) \subset \operatorname{span}(\{v_j \mid [v_i, v_j] = 0\}) + \mathfrak{z}$. Then we show $\operatorname{span}(\{v_j \mid [v_i, v_j] = 0\}) + \mathfrak{z} \subset C_{\mathfrak{g}}(\{v_i\})$.

Let $x \in C_{\mathfrak{g}}(\{v_i\})$. Write $x = \sum_j \alpha_j v_j + \sum_j \beta_j z_j$. Then,

$$\begin{aligned}
[x, v_i] &= 0 \\
\left[\sum_j \alpha_j v_j + \sum_j \beta_j z_j, v_i\right] &= 0 \\
\sum_j \alpha_j [v_j, v_i] + \sum_j \beta_j [z_j, v_i] &= 0 \\
\sum_j \alpha_j [v_j, v_i] &= 0
\end{aligned}$$

By Lemma 3.1.7, $V_i = \{[v_j, v_i] \mid [v_i, v_j] \neq 0\}$ is a linearly independent set. Then $\sum_j \alpha_j [v_j, v_i] = 0$ implies $\alpha_j = 0$ for all $1 \leq j \leq q$ such that $[v_j, v_i] \neq 0$. It follows that $x \in \operatorname{span}(\{v_j \mid [v_i, v_j] = 0\}) + \mathfrak{z}$.

Let $x \in \operatorname{span}(\{v_j \mid [v_i, v_j] = 0\}) + \mathfrak{z}$. Then,

$$\begin{aligned}[x, v_i] &= \left[\sum_j \alpha_j v_j + \sum_j \beta_j z_j, v_i\right] \\ &= \sum_j \alpha_j [v_j, v_i] + \sum_j \beta_j [z_j, v_i] \\ &= 0\end{aligned}$$

Thus $x \in C_{\mathfrak{g}}(\{v_i\})$. We conclude $C_{\mathfrak{g}}(\{v_i\}) = \operatorname{span}(\{v_j \mid [v_i, v_j] = 0\}) + \mathfrak{z}$.

Since $\mathcal{B} = V \cup Z$ is a basis of $\mathfrak{g}$ and $\{v_j \mid [v_i, v_j] = 0\} \subset V$, it follows $\{v_j \mid [v_i, v_j]0\} \cup Z$ is a linearly independent set and therefore a basis of $C_{\mathfrak{g}}(\{v_i\})$.

□

Corollary 3.1.21. Let $\mathfrak{g}$ be a uniform Lie algebra of type (p, q, r). Let $v_i \in V$. Then $s = p + q - \dim(C_{\mathfrak{g}}(\{v_i\}))$.

Proof. By Lemma 3.1.20, we know $C_{\mathfrak{g}}(\{v_i\}) = \operatorname{span}(\{v_j \mid [v_i, v_j] = 0\}) + \mathfrak{z}$ has a basis $Z \cup \{v_j \mid [v_i, v_j] = 0\}$. Since s is the number of vectors v_j such that $[v_i, v_j] \neq 0$, then $p + q - s = \dim(C_{\mathfrak{g}}(\{v_i\}))$. □

3.1.4 Classification Results of Uniform Lie Algebras

Lemma 3.1.22. Let $\mathfrak{g}$ be a Lie algebra of uniform type $(1, q, r)$. Then $q = 2r$ and $\mathfrak{g} \cong \mathfrak{h}_{2r+1}$ where $\mathfrak{h}_{2r+1}$ is the $(2r+1)$-dimensional Heisenberg algebra.

Proof. Since $p = 1$, we may write $Z = \{z_1\}$. We first show that $s = 1$. Proceed by contradiction. Assume $s > 1$. Let $v_i \in V$. Since $s > 1$, by definition of a uniform Lie algebra part 4, there exist at least two vectors $v_j, v_k \in V$ such that $[v_i, v_j] \neq 0$ and $[v_i, v_k] \neq 0$ where v_i, v_j and v_k are pairwise distinct. By definition of a uniform Lie algebra part 1, since $[v_i, v_j] \neq 0$ and $[v_i, v_k] \neq 0$, $[v_i, v_j], [v_i, v_k] \in \{\pm z_1\}$. Thus

$[v_i, v_j] = \pm[v_i, v_k]$. A contradiction arises as v_i, v_j and v_k are pairwise distinct, but by definition of a uniform Lie algebra part 3, $v_j = v_k$. Thus $s = 1$.

Define a set $R = \{\{v_i, v_j\} \mid [v_i, v_j] \neq 0\}$. Since V is non-empty, there exists $v_i \in V$; by our previous argument, there exist exactly one element $v_j \in V$ such that $[v_i, v_j] \neq 0$. Thus $\{v_i, v_j\} \in R$ and R is not empty.

Case 1: $|R| = 1$. Since there is one pair $\{v_i, v_j\} \in R$, there is exactly one pair $\{v_i, v_j\} \subset V$ such that $[v_i, v_j] \neq 0$. By definition of a uniform Lie algebra part 1, $[v_i, v_j] = \pm z_1$. By definition of a uniform Lie algebra part 3, z_1 has exactly $r = 1$ disjoint pair $\{v_i, v_j\}$ such that $[v_i, v_j] = \pm z_1$. Therefore $p = r = s = 1$. By Lemma 3.1.14, $q = 2$. Since $q = 2$, $\mathfrak{g} \cong \mathfrak{h}_3$ by Lemma 3.1.19.

Case 2: $|R| > 1$. We will show R is a partition of V. Let $\{v_i, v_j\}, \{v_l, v_m\} \in R$. Assume $\{v_i, v_j\} \bigcap \{v_l, v_m\} \neq \varnothing$. Without the loss of generality, assume $v_i = v_l$. Then $[v_i, v_j] \neq 0$ and $[v_i, v_m] \neq 0$. By definition of a uniform Lie algebra part 1, $[v_i, v_j], [v_i, v_m] \in \{\pm z_1\}$. Thus $[v_i, v_j] = \pm[v_i, v_m]$. By definition of a uniform Lie algebra part 2, $v_j = v_m$. It follows $\{v_i, v_j\} = \{v_l, v_m\}$. In particular, if $\{v_i, v_j\}, \{v_l, v_m\} \in R$ and $\{v_i, v_j\} \neq \{v_l, v_m\}$, $\{v_i, v_j\} \bigcup \{v_l, v_m\} = \varnothing$.

It is trivial to check $\bigcup_{\{v_i,v_j\}\in R}\{v_i, v_j\} \subseteq V$. We show $V \subseteq \bigcup_{\{v_i,v_j\}\in R}\{v_i, v_j\}$. Let $v_i \in V$. Since $s = 1$, there exists $v_j \in V$ such that $[v_i, v_j] \neq 0$. Therefore $\{v_i, v_j\} \in R$. In particular, $v_i \in \bigcup_{\{v_i,v_j\}\in R}\{v_i, v_j\}$ and $V \subseteq \bigcup_{\{v_i,v_j\}\in R}\{v_i, v_j\}$.

It follows $R = \{\{v_i, v_j\} \mid [v_i, v_j] \neq 0\}$ is a partition of V. Furthermore, there does not exist $v_i \in V$ such that $\{v_i\} \in R$ otherwise $[v_i, v_i] \neq 0$.

Therefore we may reorder the elements of V such that

$$[v_1', v_2'] = \cdots = [v_{q-1}', v_q'] = z_1.$$

Since R is a partition, this list has no repetitions. Thus it is clear that $\mathfrak{g}$ and $\mathfrak{h}_{2r+1}$ are isomorphic. $\square$

Corollary 3.1.23. Let $\mathfrak{g}$ be a uniform Lie algebra of type $(p, 2, r)$. Then $\mathfrak{g} \cong \mathfrak{h}_3$ where $\mathfrak{h}_3$ is the 3-dimensional Heisenberg algebra.

Proof. By Corollary 3.1.15, $p \leq \binom{2}{2} = 1$, it follows that $\mathfrak{g}$ is a uniform Lie algebra of type $(1, 2, r)$. Therefore by Lemma 3.1.22, $\mathfrak{g} \cong \mathfrak{h}_3$. $\square$

Lemma 3.1.24. Let $\mathfrak{g}$ be a uniform Lie algebra of type $(p, 3, r)$. Then $\mathfrak{g}$ is the free two-step nilpotent Lie algebra on 3 generators, $\mathfrak{f}_{3,2}$, as defined in Example 2.1.19.

Proof. Since $q = 3$, we may write $V = \{v_1, v_2, v_3\}$. We will show that $r = 1$. Proceed by contradiction. Assume $r > 1$. Let $z_l \in Z$. By definition of a uniform Lie algebra part 3, there exist at least two disjoint pairs $\{v_i, v_j\}$ and $\{v_k, v_m\}$ such that $[v_i, v_j] = \pm z_l$ and $[v_k, v_m] = \pm z_l$. Since $[\cdot, \cdot]$ is anti-commutative, $v_i \neq v_j$ and $v_k \neq v_m$. Thus v_i, v_j, v_k, v_m are pairwise distinct, contrary to the fact $q = 3$. Therefore $r = 1$.

By Lemma 3.1.14, $2p = 3s$. By Corollary 3.1.15, $p \leq \binom{3}{2} = 3$. Since p and s are integers, $p = 3$ and $s = 2$.

By definition of a uniform Lie algebra part 3, for each $z_l \in Z$ there exists exactly one pair $\{v_i, v_j\}$ such that $[v_i, v_j] = \pm z_l$. Since there are exactly $2 \cdot \binom{3}{2} = 6$ possible Lie brackets on elements of V and since $p = 3$, each bracket $[v_1, v_2] = -[v_2, v_1], [v_2, v_3] = -[v_3, v_2]$ and $[v_3, v_1] = -[v_1, v_3]$ must be nonzero. By definition of a uniform Lie algebra part 1, each nonzero bracket must be in $\{\pm z \mid z \in Z\}$. It follows we may label the elements of $\{\pm z \mid z \in Z\}$ by the bracket relations $[v_1, v_2] = -[v_2, v_1] = z_1$, $[v_2, v_3] = -[v_3, v_2] = z_2$ and $[v_3, v_1] = -[v_1, v_3] = z_3$. It becomes clear $g \cong f_{3,2}$. $\square$

Theorem 3.1.25. Let $\mathfrak{g}$ be a uniform Lie algebra of type (p, q, r) where $\dim(\mathfrak{g}) = p + q$.

1. If $\dim(\mathfrak{g}) = 3$, then $\mathfrak{g}$ is the 3-dimensional Heisenberg algebra.

2. There is no uniform Lie algebra $\mathfrak{g}$ such that $\dim(\mathfrak{g}) = 4$.

3. If $\dim(\mathfrak{g}) = p + q = 5$, then $\mathfrak{g}$ is the 5-dimensional Heisenberg algebra.

Proof. Since $\dim(\mathfrak{g}) = p + q$ we analyze all possibilities of p and q. By Lemma 3.1.17, $q \geq 2$.

1. Assume $\dim(\mathfrak{g}) = 3$. Since $p + q = 3$ and $p, q > 0$, it follows $p = 1$ and $q = 2$ since $q > 1$. By Lemma 3.1.22, $\mathfrak{g} \cong \mathfrak{h}_3$.

2. Assume $\dim(\mathfrak{g}) = 4$. Since $p + q = 4$ and $p, q > 0$ the following cases arise: $p = 1$, $q = 3$ or $p = 2$, $q = 2$.

   Case 1: Assume $p = 1$ and $q = 3$. By Lemma 3.1.24, $\mathfrak{g} \cong \mathfrak{f}_{3,2}$ contrary to $p = 1$.

   Case 2: Assume $p = 2$ and $q = 2$. Since $q = 2$, it follows by Lemma 3.1.19 that $p = 1$; a contradiction, as $p = 2$.

   Therefore there is no 4 dimensional uniform Lie algebra.

3. Assume $\dim(\mathfrak{g}) = 5$. Since $p + q = 5$ and $p, q > 0$ the following cases arise: $p = 1, q = 4$, $p = 2, q = 3$, and $p = 3, q = 2$.

   Case 1: Assume $p = 1$ and $q = 4$: By Lemma 3.1.22, $\mathfrak{g} \cong \mathfrak{h}_5$.

   Case 2: Assume $p = 2$ and $q = 3$: By Lemma 3.1.24, if $q = 3$ then $p = 3$, thus a contradiction.

   Case 3: Assume $p = 3$ and $q = 2$: By Corollary 3.1.15, we obtain $3r \leq 1$, contrary to $r \in \mathbb{N}$.

   Therefore the only possible case of a uniform Lie algebra with dimension 5 is Case 1 in which $\mathfrak{g} \cong \mathfrak{h}_5$.

□

3.2 Uniform Graphs

3.2.1 Definition and basic properties

Definition 3.2.1. Let $G = (V, E)$ be a directed (resp. undirected) graph where $|V| = q$. Let $c : E \to Z$ be a coloring function and let $p = |Z|$. Then G along with the coloring function c is a *directed (resp. undirected) uniform graph of type* (p, q, r) and degree s where $p, q, r, s > 0$, if the following conditions hold:

1. If $(v_i, v_j) \in E$, then $(v_j, v_i) \notin E$ and G is a simple graph.

2. For each color $z_l \in Z$, the color z_l is applied exactly r times.

3. Each vertex $v_i \in V$ has degree s.

Remark. When G is a directed graph, we define the degree of a vertex v_i to be the total number of adjacent vertices to v_i in Definition 2.2.2

When G is an undirected uniform graph of type (p, q, r), then G is a regular graph.

As before, we simplify our discussion of uniform graphs by using the term *graph* to refer to directed or undirected graph.

Proposition 3.2.2. Let $G = (V, E)$ be a uniform graph of type (p, q, r) with degree s. Then $sq = 2rp$ since the number of edges $|E| = rp = \frac{sq}{2}$.

Proof. By hypothesis there are q vertices each with degree s, thus the number of edges $|E| = \frac{sq}{2}$. Since each edge is colored and each color is applied to exactly r edges, $|E| = rp$. It follows that $rp = |E| = \frac{sq}{2}$ and we obtain $sq = 2rp$. □

Remark. By Proposition 3.2.2, we may exclude the degree s from the definition of a uniform graph as $s = \frac{2rp}{q}$. Thus when defining a uniform graph we will not explicitly define s as it will be implicitly defined by (p, q, r).

Corollary 3.2.3. Let $G = (V, E)$ be a uniform graph of type (p, q, r). The number of colors p divides the number of edges $|E|$.

Proof. By Proposition 3.2.2, the number of edges $|E| = rp$. Thus $p \mid |E|$. $\square$

Lemma 3.2.4. Let G be a uniform graph of type (p, q, r). Then the number of colors, p is bounded by $s \leq p \leq |E|$, where s is the degree of each vertex and $|E| = \frac{sq}{2} = rp$ is the number of edges.

Proof. Since no two adjacent edges may share the same color and each vertex is of total degree s, it follows there must be at least s colors to construct a valid uniform coloring on G. Therefore it follows $s \leq p$.

Each edge must have a color, and each color must appear at least once; it follows that $p \leq |E|$. Otherwise, if $p > |E|$ a contradiction would arise, as at most $|E|$ colors would be applied and the remaining colors would not be applied. $\square$

Remark. Let $G = (V, E)$ be a uniform graph of type (p, q, r) with an edge coloring $c : E \to Z$. We summarize the descriptions of p, q, r and s in terms of the graph

r : the number of times a particular color is applied to an edge.

$p = |Z|$: the number of colors of the edge coloring.

s : the total degree at each vertex.

$q = |V|$: the number of vertices.

$rp = \frac{sq}{2} = |E|$: the number of edges.

In the next example we show how to associate a uniform Lie algebra of type (p, q, r) to a uniform graph of type (p, q, r).

Example 3.2.5. Let $\mathfrak{h}_7$ be the seven dimensional Heisenberg algebra with uniform basis $\mathcal{B} = \{v_1, ..., v_6\} \cup \{z_1\}$ of type $(1, 6, 3)$, as defined in Example 3.1.2. Then $\mathfrak{h}_7$ has Lie brackets

$$[v_1, v_2] = [v_3, v_4] = [v_5, v_6] = z_1.$$

Define a directed graph $G = (V, E)$ where $V = \{v_1, v_2, v_3, v_4, v_5, v_6\}$ and $E = \{(v_1, v_2), (v_3, v_4), (v_5, v_6)\}$. Let $c : E \to Z$ be defined by $c(v_i, v_j) = [v_i, v_j] = z_1$ for all $(v_i, v_j) \in E$. Then c is an edge coloring function. Then G, along with the edge coloring function c, defines a directed uniform graph of type $(1, 6, 3)$, as in Figure 3.1.

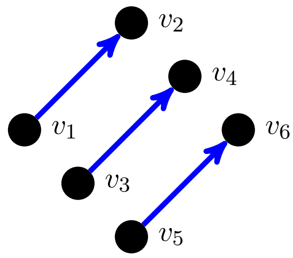


Figure 3.1: Uniform graph of type $(1, 6, 3)$

The following theorem extends Example 3.2.5 to show that every uniform Lie algebra of type (p, q, r) gives rise to a corresponding uniform graph of the same type.

Theorem 3.2.6. Let $\mathfrak{g}$ be a uniform Lie algebra of type (p, q, r) with uniform basis $\mathcal{B} = V \cup Z$. Then $\mathfrak{g}$ with $\mathcal{B}$ defines a directed uniform graph of type (p, q, r).

Proof. Define a graph $G = (V, E)$ where $V = \{v_1, \dots, v_q\}$ defined by the basis $\mathcal{B} = V \cup Z$ of $\mathfrak{g}$ and $E = \{(v_i, v_j) \mid [v_i, v_j] \in Z\}$. Define a function $c : E \to Z$ by $c(v_i, v_j) = [v_i, v_j] = z_l \in Z$. By construction of E, $c(E) = Z$, and thus c is surjective. Assume edges $e_1, e_2 \in E$ are adjacent such that $c(e_1) = c(e_2)$. We may write $e_1 = (v_i, v_j)$ where $v_i, v_j \in V$. By definition of c, $c(e_1) = c(v_i, v_j) = [v_i, v_j] = z_l \in Z$. Since e_2 is adjacent to e_1, $e_2 = (v_i, v_k)$ or (v_k, v_i) for some $v_k \in V$; regardless of choice, by definition of c, $[v_i, v_k] = \pm z_l$. Then by part 2 of the definition of a uniform Lie algebra, $[v_i, v_j] = \pm[v_i, v_k]$, and $v_j = v_k$. Therefore, there are no adjacent edges with the same color, and c is a coloring function.

We have now shown $G = (V, E)$ is a directed graph with a coloring function $c : E \to Z$. We will show further that G is a uniform graph of type (p, q, r) where p, q, r are given by the type of $\mathfrak{g}$.

1. Let $(v_i, v_j) \in E$. Then $[v_i, v_j] \in Z$ and $[v_j, v_i] = -[v_i, v_j]$. Thus $[v_j, v_i] \notin Z$ and $(v_j, v_i) \notin E$. Furthermore, since $[v_i, v_i] = 0$ for all $v_i \in V$, $(v_i, v_i) \notin E$. Thus G is simple.

2. Let $z_l \in Z$. Then by definition of a uniform Lie algebra there exist exactly r disjoint pairs $\{v_i, v_j\}$ such that $[v_i, v_j] = \pm z_l$. Therefore, there are exactly r edges (v_i, v_j) such that $c(v_i, v_j) = z_l$.

3. Let $v_i \in V$. By definition of a uniform Lie algebra, there exist exactly s vectors v_j such that $[v_i, v_j] \neq 0$. Thus, there exist s edges incident to v_i, and v_i has degree s.

Therefore $G = (V, E)$ along with edge coloring function c defines a uniform graph of type (p, q, r). □

Example 3.2.7. Let $\mathfrak{g}$ be the Lie algebra defined in Example 3.1.4. In particular, $\mathfrak{g}$ has a uniform basis $\mathcal{B} = \{v_1, v_2, v_3, v_4\} \cup \{z_1, z_2\}$ and Lie brackets $[v_1, v_2] = z_1$ and $[v_3, v_4] = z_2$. By Theorem 3.2.6, $G = (V, E)$ defines a directed graph where $V = \{v_1, v_2, v_3, v_4\}$ and $E = \{(v_1, v_2), (v_3, v_4)\}$. The coloring function $c : E \to Z$ defined by $c(v_1, v_2) = z_1$ and $c(v_3, v_4) = z_2$ along with the graph G defines a directed uniform graph of type $(2, 4, 1)$ corresponding to $\mathfrak{g}$. In Figure 3.2 we present G.

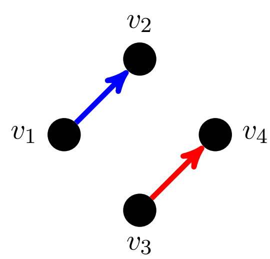


Figure 3.2: Uniform graph of type $(2, 4, 1)$

Example 3.2.8. Let $\mathfrak{h}$ be the Lie algebra defined in Example 3.1.5. In particular $\mathfrak{g}$ has uniform basis $\mathcal{B} = \{v_1, v_2, v_3, v_4\} \cup \{z_1, z_2\}$ with Lie brackets $[v_1, v_2] = [v_3, v_4] = z_1$ and $[v_1, v_3] = [v_2, v_4] = z_2$. By Theorem 3.2.6, $H = (V, E)$ defines a directed graph where

$V = \{v_1, v_2, v_3, v_4\}$ and $E = \{(v_1, v_2), (v_2, v_3), (v_3, v_4), (v_4, v_1)\}$. The coloring function $c : E \to Z$ defined by $c(v_1, v_2) = c(v_3, v_4) = z_1$ and $c(v_2, v_3) = c(v_4, v_1) = z_2$, along with the graph H, defines a uniform graph of type $(2, 4, 2)$ corresponding to $\mathfrak{h}$, as seen in Figure 3.3.

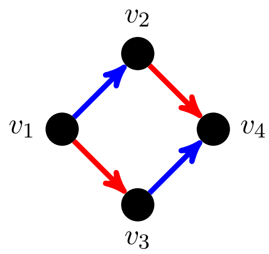


Figure 3.3: Uniform graph of type $(2, 4, 2)$

Conversely, given a uniform graph we may define a uniform Lie algebra of the same type. The following example shows how a Lie algebra may be constructed from a uniform graph.

Example 3.2.9. Let $G = (V, E)$ be a directed graph where $V = \{v_1, v_2, v_3\}$ and let $E = \{(v_1, v_2), (v_2, v_3), (v_3, v_1)\}$. Let $c : E \to Z = \{z_1, z_2, z_3\}$ be a coloring function where $c(v_1, v_2) = z_1$, $c(v_2, v_3) = z_2$, and $c(v_3, v_1) = z_3$. Then $G = (V, E)$ along with the coloring function c defines a uniform graph of type $(3, 3, 1)$, as represented in Figure 3.4.

Let $\mathfrak{g}$ be a six-dimensional vector space and identify a basis $\mathcal{B}$ with $V \cup Z$. Define a Lie bracket on this vector space by

$$[v_1, v_2] = c(v_1, v_2) = z_1 \qquad [v_2, v_3] = c(v_2, v_3) = z_2 \qquad [v_3, v_1] = c(v_3, v_1) = z_3.$$

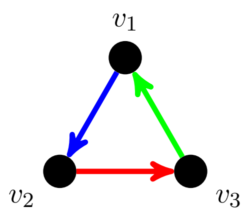


Figure 3.4: Uniform graph of type (3,3,1)

Then $\mathfrak{g}$ is the free two-step nilpotent Lie algebra on three generators as defined in Example 2.1.18.

To generalize the process given in Example 3.2.9, the following theorem proves that every uniform graph of type (p, q, r) gives rise to a uniform Lie algebra of the same type.

Theorem 3.2.10. Let $G = (V, E)$ be a directed uniform graph of type (p, q, r) with coloring function $c : E \to Z$. Let $\mathfrak{g}$ be a vector space with basis $\mathcal{B}$ identified by $V \cup Z$. Then G defines a uniform Lie algebra of type (p, q, r) on $\mathfrak{g}$.

Proof. Define a Lie bracket $[\cdot, \cdot] : \mathfrak{g} \times \mathfrak{g} \to \mathfrak{g}$ on the basis $\mathcal{B} = V \cup Z$ of $\mathfrak{g}$ by

$$[x_i, x_j] = \begin{cases} c(x_i, x_j) & \text{if } (x_i, x_j) \in E \\ -c(x_j, x_i) & \text{if } (x_j, x_i) \in E \\ 0 & \text{otherwise} \end{cases}$$

and extend bilinearly. Note: Since G is a uniform graph, if $(v_i, v_j) \in E$, then $(v_j, v_i) \notin E$ and thus there is no ambiguity in our definition.

The vector space $\mathfrak{g}$ over $\mathbb{F}$ on a basis $\mathcal{B} = \{v_1, \ldots, v_q\} \cup \{z_1, \ldots, z_p\}$ has the property that $[v_i, v_j] = -[v_j, v_i]$ for any distinct $v_i, v_j \in V$ and $[v_i, v_i] = 0$ for any $v_i \in V$. Furthermore, $[z_i, z_k] = 0$ for all $z_i, z_k \in Z$. Then $[x, x] = 0$ for any $x \in \mathfrak{g}$.

Let $x_i, x_j, x_k \in \mathcal{B}$. If $[x_j, x_k] = 0$ then $[x_i, [x_j, x_k]] = [x_i, 0] = 0$; otherwise, if $[x_j, x_k] \neq 0$ it follows $[x_i, [x_j, x_k]] = [x_i, \pm z_l] = 0$. Therefore $[\cdot, \cdot]$ satisfies the Jacobi identity trivially and $\mathfrak{g}$ is a Lie algebra.

We now show $\mathfrak{g}$ with basis $\mathcal{B}$ is a uniform Lie algebra:

1. Let $v_i, v_j \in V$. Then $[v_i, v_j] \in \{0, \pm z_1, ..., \pm z_p\}$ for all $v_i, v_j \in V$ by construction of $[\cdot, \cdot]$.

2. Let $v_i, v_j \in V$. Assume $[v_i, v_j] \neq 0$ and $[v_i, v_j] = \pm[v_i, v_k]$ for some $v_k \in V$. Then either $(v_i, v_j) \in E$ or $(v_j, v_i) \in E$. Similarly, since $[v_i, v_k] \neq 0$ either $(v_i, v_k) \in E$ or $(v_k, v_i) \in E$. Since $[v_i, v_j] = \pm[v_i, v_k]$, the corresponding edges are colored the same. Since c is a coloring function, these edges cannot be adjacent, thus it follows that $v_j = v_k$.

3. Let $z_l \in Z$. By definition of a uniform graph there are exactly r edges $(v_i, v_j) \in E$ such that $c(v_i, v_j) = z_l$. Since G is a simple graph, there are exactly r edges (v_i, v_j) colored with z_l. Thus there are r disjoint pairs $\{v_i, v_j\}$ such that $[v_i, v_j] = \pm z_l$.

4. Let $v_i \in V$. As G is a uniform graph of type (p, q, r), each vertex has degree s. Thus there are exactly s vertices v_j such that (v_i, v_j) or $(v_j, v_i) \in E$. Then it follows there are s basis vectors v_j such that $[v_i, v_j] \neq 0$.

Therefore $\mathfrak{g}$ is a uniform Lie algebra of type (p, q, r). $\square$

Corollary 3.2.11. Assume $\mathbb{F} = \mathbb{C}$. Let $\mathfrak{g} = \mathfrak{v} \oplus \mathfrak{z}$ be a directed uniform Lie algebra of type (p, q, r) with uniform basis $\mathcal{B} = V \cup Z$. Then by Theorem 3.2.6, $\mathfrak{g}$ with basis $\mathcal{B}$ defines a uniform graph $G = (V, E)$ of type (p, q, r). Endow $\mathfrak{g}$ with the natural inner product $\langle \cdot, \cdot \rangle$. Let $z_l \in Z$ and let J_{-z_l} be Kaplan's J_z map defined in Definition 2.1.16. Let $H = (V', E')$ be the subgraph of G where $V' = V$ and $E' = \{e \in E \mid c(e) = z_l\}$. Then the linear map, J_{-z_l} written in matrix form encodes the adjacency matrix of the subgraph H of $G = (V, E)$. Let $[J_{-z_l}]$ denote the matrix for J_{-z_l} with respect to the basis V of $\mathfrak{v}$. Then the non-negative entries of $[J_{-z_l}]$ encode the adjacency matrix A of the subgraph H.

Proof. Let z_l be fixed. By construction of the uniform graph G with coloring function $c : E \to Z$, $E = \{(v_i, v_j) \mid [v_i, v_j] \in Z\}$ and $c(v_i, v_j) = [v_i, v_j]$. The adjacency matrix A of the subgraph $H = (V', E')$ is defined by

$$A_{ij} = \begin{cases} 1 & \text{if } (v_i, v_j) \in E' \\ 0 & \text{otherwise.} \end{cases}$$

Let $(v_i, v_j) \in E' = \{e \in E \mid c(e) = z_l\} \subset E$, then $[v_i, v_j] \neq 0$ and $c(v_i, v_j) = z_l$. In particular, the (i, j) entry of the adjacency matrix A of H is given by 1 if $[v_i, v_j] = z_l$ and 0 otherwise.

Let $[J_{-z_l}]$ denote the matrix for J_{-z_l} with respect to the basis V of $\mathfrak{v}$. Then

$$[J_{-z_l}]_{ij} = \begin{cases} 1 & \text{if } [v_i, v_j] = z_l \\ -1 & \text{if } [v_j, v_i] = -z_l \\ 0 & \text{otherwise.} \end{cases}$$

In particular, if we restrict this matrix to only positive entries (and require all negative entries be zero), the (i, j) entry is given by 1 if $[v_i, v_j] = z_l$ and zero otherwise; exactly the form for the adjacency matrix A of H.

$\square$

Example 3.2.12. Let $G = (V, E)$ with $c : E \to Z = \{z_1, z_2, z_3\}$ be the uniform graph defined in Example 3.2.9. Then the subgraph H defined by the color z_1, shown in Figure 3.5, has adjacency matrix

$$A = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

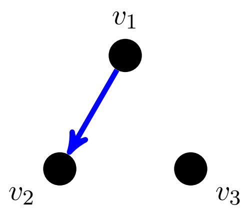


Figure 3.5: The subgraph of G defined by z_1.

Furthermore, the Kaplan J_z from Definition 2.1.16 $-z_1$ in matrix form is

$$J_{-z_1} = \begin{pmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

It follows the non-negative entries of J_{-z_1} define the adjacency matrix on z_1 as stated in Corollary 3.2.11.

Definition 3.2.13. Let G be a directed uniform graph of type (p, q, r). Let $\mathfrak{g}$ be the uniform Lie algebra induced by G as in Theorem 3.2.10. Then we call $\mathfrak{g}$ the *uniform Lie algebra corresponding to* G.

3.2.2 Graph Constructions

In this section we analyze methods of general constructions of uniform graphs.

Example 3.2.14. Let $G = (V, E)$ be an s-regular graph. Let $p = |E|$ and let $q = |V|$. Write $E = \{e_1, \ldots, e_p\}$. Define a coloring function $c : E \to Z$ by $c(e_i) = z_i$. Clearly c is surjective. Since each edge is colored uniquely, no two adjacent edges may share the same color. Thus G is a uniform graph of type (p, q, r).

Proposition 3.2.15. Let $G = (V, E)$ be the directed cycle graph with $q > 2$ vertices as in Figure 3.6. Let $Z = \{z_0, ..., z_{p-1}\}$ where $p > 1$. Then there exists a coloring function $c : E \to Z$ which makes G a uniform graph of type (p, q, r) if and only if $rp = q$.

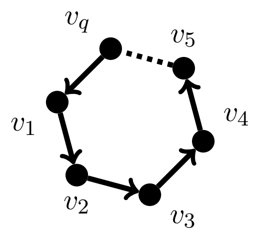


Figure 3.6: Directed cycle graph with $q > 2$ vertices

Proof. Assume G with a color function $c : E \to Z$ is a uniform graph of type (p, q, r). By Proposition 3.2.2, $2rp = sq$. Then since $s = 2$ it follows $rp = q$.

Conversely, assume $rp = q$. Label the edges as $E = \{e_0, \ldots, e_{q-1}\}$ where edges e_i and e_{i+1} for all $0 \leq i \leq q-1$ are adjacent and edges e_0 and e_{q-1} are adjacent. Define a coloring function $c : E \to Z$ by $c(e_i) = z_{i \mod p}$. Since $q > p$, c is a surjective function.

Let $e_i, e_j \in E$ be adjacent edges. Without the loss of generality, assume $i < j$. Then $j = i + 1$ or $i = 0$ and $j = q - 1$. If $j = i + 1$, then $i \not\equiv (i + 1) \mod p$. In particular $c(e_i) \neq c(e_j)$. If $i = 0$ and $j = q - 1$, then $j \equiv (q - 1) \mod p$; by hypothesis $q = rp$, and $j \equiv -1 \mod p$. Since $p > 1$, $c(e_i) \neq c(e_j)$. It follows no two adjacent edges share the same color.

Let $z_l \in Z$. By congruence modulo p on all integers $0 \leq j \leq q - 1$ and hypothesis that $rp = q$, there are exactly r integers $\{kp + l \mid 0 \leq k \leq r\}$ where each $kp + l \equiv l \mod p$. Thus the edge coloring function c colors exactly r edges with color z_l.

Therefore G with the coloring function c is a uniform graph of type (p, q, r). $\square$

The following theorems relate how two disjoint uniform graphs may be used to construct new uniform graphs. In the first theorem we combine disjoint uniform graphs which share the same color set and in the second theorem we combine uniform graphs which have disjoint color sets.

Theorem 3.2.16. Let $G_1 = (V_1, E_1)$ be a uniform graph of type (p_1, q_1, r_1) and $G_2 = (V_2, E_2)$ be a uniform graph of type (p_2, q_2, r_2) sharing the color set Z. Assume further $V_1 \cap V_2 = \varnothing$. If each vertex of G_1 and G_2 has degree s then $G = G_1 \cup G_2$ is a uniform graph of type $(p, q_1 + q_2, r_1 + r_2)$, where $p = p_1 = p_2$.

Proof. Since each vertex of G_1 and G_2 has degree s, it follows $G = G_1 \cup G_2$ is a graph where each vertex is of degree s.

Let $c_1 : E_1 \to Z$ be the coloring function for the uniform graph G_1. Let $c_2 : E_2 \to Z$ be the coloring function for the uniform graph G_2. Then define a function $c : E_1 \cup E_2 \to Z$ for $G = (V_1 \cup V_2, E_1 \cup E_2)$ by

$$c(e) = \begin{cases} c_1(e) & \text{if } e \in E_1 \\ c_2(e) & \text{if } e \in E_2 \end{cases}.$$

Then c is a surjection onto Z. Furthermore, since E_1 and E_2 are disjoint, no edge from E_1 is adjacent with any edge from E_2. Since c_1 and c_2 are coloring functions, all adjacent edges must be colored differently by the function c. Thus c is a color function on G. Furthermore, c_1 applies each color to exactly r_1 edges and c_2 applies each color to exactly r_2 edges; it follows that c applies each color to exactly $r_1 + r_2$ edges. Therefore G is a uniform graph of type $(p, q_1 + q_2, r_1 + r_2)$. $\square$

Theorem 3.2.17. Let $G_1 = (V_1, E_1)$ be a uniform graph of type (p_1, q_1, r_1) with coloring function $c_1 : E_1 \to Z_1$ and let $G_2 = (V_2, E_2)$ be a uniform graph of type (p_2, q_2, r_2) with coloring function $c_2 : E_2 \to Z_2$. Suppose $Z_1 \cap Z_2 = \varnothing$ and G_1 and G_2 are disjoint graphs. Define a graph $G = G_1 \cup G_2$ and coloring function $c : E \to Z$ by

$$c(e) = \begin{cases} c_1(e) & \text{if } e \in E_1 \\ c_2(e) & \text{if } e \in E_2 \end{cases}.$$

Then G is a uniform graph of type $(p_1 + p_2, q_1 + q_2, r)$ and with each vertex of degree s if and only if $r = r_1 = r_2$ and each vertex of G_1 and G_2 has degree s.

Proof. Suppose G is a uniform graph of type $(p_1 + p_2, q_1 + q_2, r)$ and each vertex of degree s. Since G_1 and G_2 are disjoint graphs, $E_1 \cap E_2 = \varnothing$. In particular, there are no edges which connect a vertex of G_1 to a vertex of G_2. It follows each vertex of V_1 and each vertex of V_2 must be of degree s, and $s = s_1 = s_2$.

Let $z_l \in Z_1 \cup Z_2$. Since Z_1 and Z_2 are disjoint, $z_l \in Z_1$ or $z_l \in Z_2$, but not both. Without the loss of generality, let $z_l \in Z_1$. By definition of a uniform graph on G, the color z_l must appear exactly r times. We have already observed there are no edges joining a vertex of V_1 to a vertex of V_2. Since $z_l \in Z_1$, the color z_l may not appear on an edge in E_2. Therefore, the color z_l must appear on exactly r edges of E_1. It follows $r = r_1$. When $z_l \in Z_2$, it follows similarly that $r = r_2$. Therefore, $r = r_1 = r_2$.

Conversely, assume the vertices of G_1 and G_2 have degree s and $r_1 = r_2$. To show G is a uniform graph, we show that each vertex of G has degree s, c is a surjective function, then show c is a coloring function by showing no two edges adjacent edges of G share the same color. Since G_1 and G_2 are disjoint, there are no edges connecting a vertex of G_1 to vertex of G_2. It follows each vertex of G has degree s. Since c_1 and c_2 are surjective functions on E_1 and E_2 respectively and since $E = E_1 \cup E_2$, $c(E) = c(E_1) \cup c(E_2) = Z_1 \cup Z_2$. Therefore c is a surjective function onto $Z = Z_1 \cup Z_2$.

Let $e_i, e_j \in E$ be adjacent edges of G. Since G_1 and G_2 are disjoint, e_i, e_j must be edges of G_1 or G_2, but not both. Without the loss of generality, assume $e_i, e_j \in E_1$. Then $c(e_i) = c_1(e_i)$ and $c(e_j) = c_1(e_j)$. Since G_1 is a uniform graph and e_i, e_j are adjacent, $c_1(e_i) \neq c_1(e_j)$. Similarly, if $e_i, e_J \in E_2$, $c_2(e_i) \neq c_2(e_j)$. It follows, $c(e_i) \neq c(e_j)$. Thus, no two adjacent edges of E share the same color. Therefore $G = G_1 \cup G_2$ is a uniform graph of type $(p_1 + p_2, q_1 + q_2, r)$ where $r = r_1 = r_2$. □

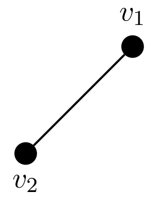
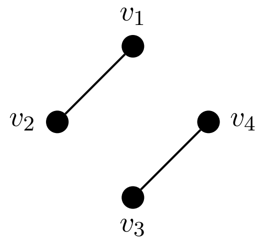
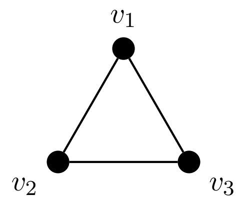
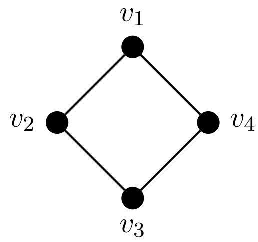
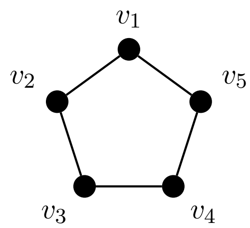
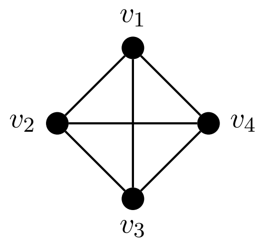
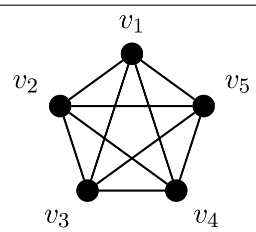
$s\backslash\lvert V\rvert$	2	3	4	5
1				
2				
3				
4				

Table 3.1: All s-regular graphs with five or fewer vertices (up to isomorphism)

3.2.3 Graph Classification

In this section we classify undirected uniform graphs up to edge-color isomorphism. Our main result Theorem 3.2.18 proves all undirected uniform graphs with five or fewer vertices are listed (up to edge-color isomorphism) in Table 3.2.

Theorem 3.2.18. If $G = (V, E)$ is an undirected uniform graph of type (p, q, r) with $|V| \leq 5$, then G occurs exactly once among the graphs in Table 3.2.

Remark. Among the graphs appearing in Table 3.2, each undirected uniform graph has a unique value for (p, q, r). Therefore we name each graph $G_{(p,q,r)}$ where (p, q, r) refers to the uniform graph type.

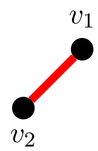
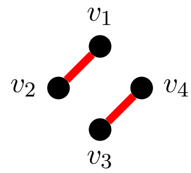
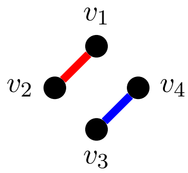
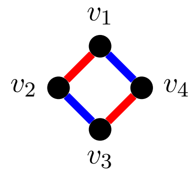
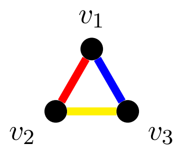
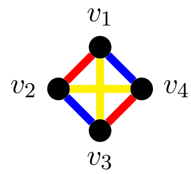
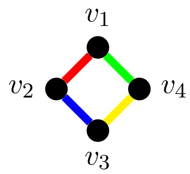
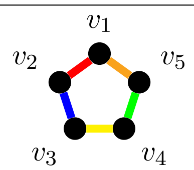
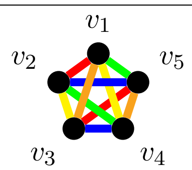
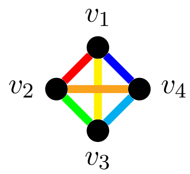
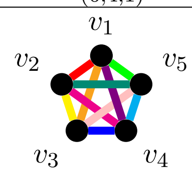
Number of colors		
1	 $G_{(1,2,1)}$	 $G_{(1,4,2)}$
2	 $G_{(2,4,1)}$	 $G_{(2,4,2)}$
3	 $G_{(3,3,1)}$	 $G_{(3,4,2)}$
4	 $G_{(4,4,1)}$	
5	 $G_{(5,5,1)}$	 $G_{(5,5,2)}$
6	 $G_{(6,4,1)}$	
10	 $G_{(10,5,1)}$	

Table 3.2: All undirected uniform graphs with 5 or fewer vertices (up to edge-color isomorphism).

This naming convention is not possible when $q = |V| \geq 6$ as there are non-isomorphic uniform graphs with the same uniform graph type.

Remark. We have compared each graph in Table 3.2 against previous classifications of dimension seven and less found in [7, 15, 25]. In particular, for each graph, we verified, for dimensions seven and less, each one of these graphs appear in these classification. Since less is known of two-step nilpotent Lie algebras of dimension 8 and greater, some of the graphs contained in the Table 3.2 cannot be compared with previous classifications.

Proof of Theorem 3.2.18. By the definition of an undirected uniform graph, we know the underlying graph is regular. These graphs with five and fewer vertices appear in Table 3.1. In all cases the number of colors p is constrained by the conditions from Lemma 3.2.4

$$s \leq p \leq |E| \tag{3.2}$$

and by Corollary 3.2.3

$$p \mid |E| \tag{3.3}$$

where $|E|$ is the number of edges and s is the degree on each vertex. We proceed by cases on the number of vertices $|V|$.

Case 1: $|V| = 2$

In Table 3.1, we see there is exactly one regular graph with two vertices. Since there is only one edge to color, the graph must be isomorphic to $G_{(1,2,1)}$.

Case 2: $|V| = 3$

In Table 3.1, there is exactly one regular graph with three vertices. By Equation 3.2 the number of colors is bounded by $2 \leq p \leq 3$ and by Equation 3.3 the number of colors is constrained by $p \mid 3$. Therefore it follows $p = 3$ and have $G \cong G_{(3,3,1)}$.

Case 3: $|V| = 4$

By Table 3.1, there are three regular graphs with four vertices, one with two edges, one with four edges, and one with six edges. We consider each case separately.

Case 3.A: If we have the graph in Table 3.1 with two edges and four vertices, then by Equation 3.2 and Equation 3.3 it follows that $1 \leq p \leq 2$ and $p \mid 2$. If $p = 1$ clearly $G \cong G_{(1,4,2)}$. If $p = 2$, it is clear, $G \cong G_{(2,4,1)}$.

Case 3.B: If we have the graph in Table 3.1 with four edges and four vertices, from Equation 3.2 and Equation 3.3 it follows that $2 \leq p \leq 4$ and $p \mid 4$. Hence $p = 2$ or $p = 4$. Assume $p = 2$ and choose a coloring for one edge. The graph is determined as adjacent edges may not share the same color. It becomes clear, $G \cong G_{(2,4,2)}$. If $p = 4$, each edge is colored with a different color and $G \cong G_{(4,4,1)}$.

Case 3.C: If we have the graph in Table 3.1 with six edges and four vertices, then by Equation 3.2 and Equation 3.3 then $3 \leq p \leq 6$ and $p \mid 6$. Thus $p = 3$ or $p = 6$. If $p = 3$, we fix a vertex and color the three edges incident to this vertex with three different colors. Since adjacent edges may not share the same color, the remaining colors are determined. Thus $G \cong G_{(3,4,2)}$. If $p = 6$ each edge is colored with a different color and it is clear that $G \cong G_{(6,4,1)}$.

Case 4: $|V| = 5$

In Table 3.1, there are two regular graphs with five vertices, one with five edges and one with ten edges.

If G has five vertices and five edges then by Equation 3.2 the number of colors

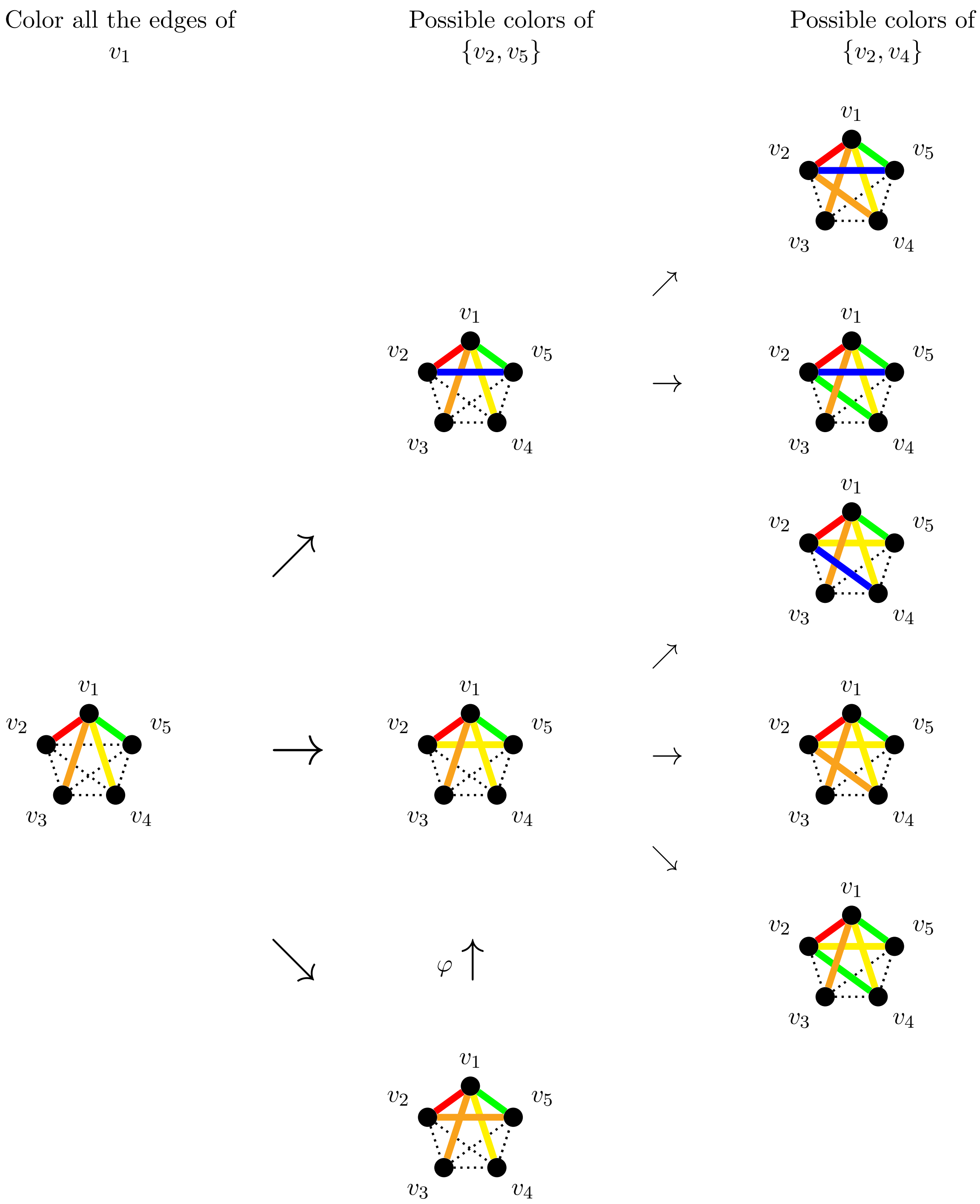


Figure 3.7: The process of enumerating all possible uniform colorings of K_5.

p is bounded by $2 \leq p \leq 5$. Moreover by Equation 3.3, the number of colors must divide the number of edges. It follows that $p = 5$. Then each edge must have a different color and $G \cong G_{(5,5,1)}$.

If G has five vertices and ten edges then by Equation 3.2, the number of colors p is bounded by $4 \leq p \leq 10$, and by Equation 3.3, $p \mid 10$. Thus $p = 5$ or $p = 10$.

Suppose $p = 10$. Then $G \cong G_{(10,5,1)}$ and it is clear that every undirected uniform graph with five vertices, ten edges, and ten colors must be isomorphic to $G_{(10,5,1)}$.

Suppose $p = 5$. To prove this case, we will enumerate all possible colorings for G. Let $V = \{v_1, ..., v_5\}$ and $Z = \{c_1, ..., c_5\}$. The general outline for this process is to fix a coloring for the four incident edges to v_1, then proceed to color all edges of v_2 until we have determined an edge coloring for G. This process is illustrated in Figure 3.7. In this and subsequent figures, uncolored edges are represented by a dotted black edge, c_1 is blue, c_2 is red, c_3 is orange, c_4 is yellow, and c_5 is green. Since adjacent edges cannot be colored with the same color, without the loss of generality, we color the four edges incident to v_1 by

$$c(\{v_1, v_2\}) = c_2, \quad c(\{v_1, v_3\}) = c_3, \quad c(\{v_1, v_4\}) = c_4, \quad \text{and} \quad c(\{v_1, v_5\}) = c_5$$

as shown as the first step in Figure 3.7. Next we move to vertex v_2 and proceed to color all edges incident to v_2. Since we have already colored adjacent edges by $c(\{v_1, v_2\}) = c_2$ and $c(\{v_1, v_5\}) = c_5$, we may use only one of the colors c_1, c_3 or c_4 on the edge $\{v_2, v_5\}$. These three cases are illustrated in Figure 3.7.

By Lemma 3.1.14, $2rp = sq$. We find that

$$r = \frac{sq}{2p} = \frac{4 \cdot 5}{2 \cdot 5} = 2. \tag{3.4}$$

Thus each color must appear exactly two times. We then analyze each case which arises from a choice of color on the edge $\{v_2, v_5\}$:

(a) Let $c(\{v_2, v_5\}) = c_1$. Since edges incident to v_2 are already colored it follows the only possible colors for $\{v_2, v_4\}$ are c_2 and c_5.

(b) Let $c(\{v_2, v_5\}) = c_4$. Similarly, edges incident to v_2 have been assigned colors, we see the only possible colors for $\{v_2, v_4\}$ are c_1, c_3 and c_5.

(c) Let $c(\{v_2, v_5\}) = c_3$. The edges incident to v_2 are already colored, the only possible colors for $\{v_2, v_4\}$ are c_1 and c_5. Then the graph isomorphism $\varphi = (2\ 5)(3\ 4)$ along with a relabeling of colors shows the graphs of (c) and (b) are isomorphic as edge colored graphs, as shown in Figure 3.8.

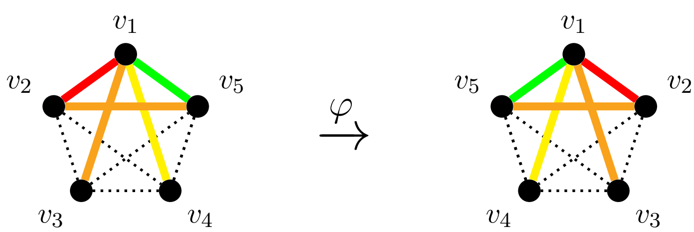


Figure 3.8: Applying the graph isomorphism φ to the colored graph of (c).

There are five cases: two from (a) and three from (b). Each case is shown in Figure 3.7. We consider each case separately.

Case 4.A: Assume $c(\{v_2, v_5\}) = c_1$ and $c(\{v_2, v_4\}) = c_3$.

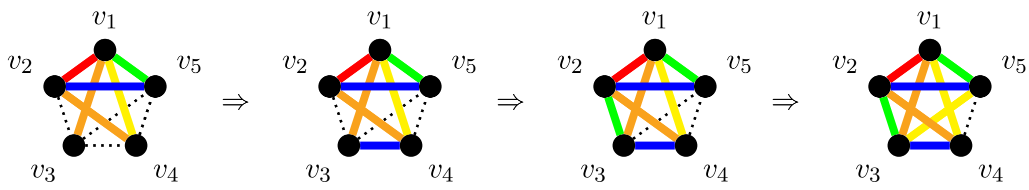


Figure 3.9: The enumeration procedure as described in Case 4A.

We can see this graph as in the upper right hand corner of Figure 3.7. By Equation 3.4, we know $r = 2$; in particular, we may only apply each color twice. Since the color c_3 (orange) has already been applied twice, the remaining edges must be colored only with c_1, c_2, c_4 and c_5. This process is illustrated in Figure 3.9.

We analyze all possible edges in which c_1 (blue), may be applied to. Since we know adjacent edges may not share the same color and there are 4 uncolored edges, 3 of which are adjacent to $\{v_2, v_5\}$. It follows c_1 may only be applied to the edge $\{v_3, v_4\}$. Then the remaining three edges must be colored with c_2, c_4 and c_5.

In a similar way we analyze where the color c_5 (green) may be applied. The only possibility for the color c_5 is the edge $\{v_2, v_3\}$ as there are three remaining edges to be colored, but two of these edges have an adjacent edge colored by c_5. There are two uncolored edges remaining, $\{v_3, v_5\}$ and $\{v_4, v_5\}$. The edge $\{v_4, v_5\}$ cannot be colored by c_4 (yellow), so $c(\{v_4, v_5\}) = c_2$ (red). Finally, $\{v_3, v_5\}$ must be colored by c_4 in order for c_4 to occur twice as illustrated in Figure 3.9.

Then we apply the graph isomorphism $\varphi(v_j) = v_{\sigma(j)}$ where $\sigma =$

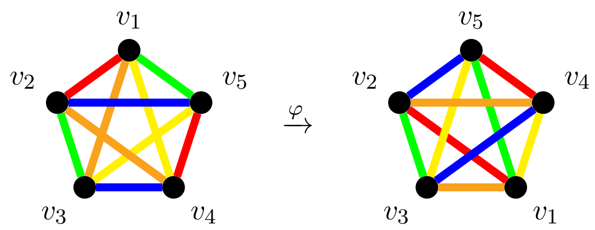


Figure 3.10: Applying graph isomorphism φ to the colored graph of Case 4A

$(1,\ 5,\ 4) \in S_5$ as in Figure 3.10.This graph is clearly isomorphic to $G_{(5,5,2)}$ as in Table 3.2.

Case 4.B: Assume $c(\{v_2, v_5\}) = c_1$ and $c(\{v_2, v_4\}) = c_5$.

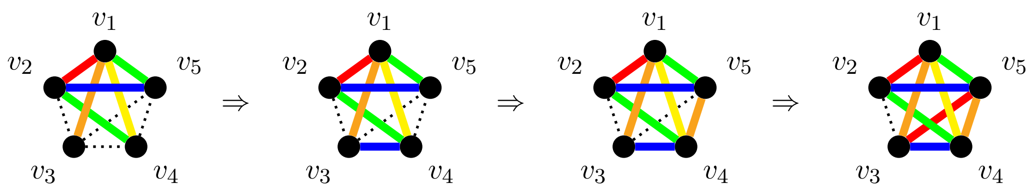


Figure 3.11: The enumeration procedure as described in Case 4B.

We have applied the color c_5 (green) twice, and cannot apply it to any more edges. Thus we must color the 4 remaining edges with one of the remaining colors c_1, c_2, c_3, and c_4. We see that the only valid edge for c_1 (blue) to be applied to is $\{v_3, v_4\}$ as all other remaining edges are adjacent to $\{v_2, v_5\}$ which is already colored with c_1.

The only remaining uncolored edges are $\{v_2, v_3\}$, $\{v_3, v_5\}$ and $\{v_4, v_5\}$ and the only remaining colors to be applied are c_2, c_3, and c_4. Since an edge incident to v_3 is already colored by c_3 (orange), no other edge incident to v_3 may have color c_3; therefore c_3 must be applied to $\{v_4, v_5\}$.

The edges $\{v_2, v_3\}$ and $\{v_3, v_5\}$ remain uncolored, and the colors c_2 and c_4 must still be applied. Then c_2 must be applied to $\{v_3, v_5\}$, leaving $c(\{v_2, v_3\}) = c_4$.

It follows G is the graph given by Figure 3.12 and clearly $G \cong G_{(5,5,2)}$.

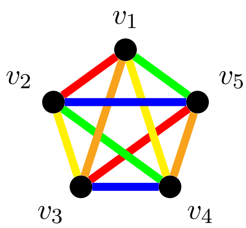


Figure 3.12: The colored graph of Case 4B

Case 4.C: Assume $c(\{v_2, v_5\}) = c_4$ and $c(\{v_2, v_4\}) = c_1$.

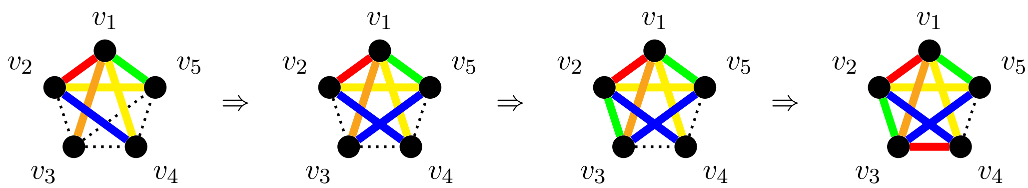


Figure 3.13: Illustrating the enumeration procedure as described in Case 4C.

We must apply the colors c_1, c_2, c_3 and c_5 to the remaining four uncolored edges. Since adjacent edges may not share the same color, the color c_1 (blue) may only be applied to the edge $\{v_3, v_5\}$, as all of the other remaining uncolored edges are adjacent to the edge $\{v_2, v_4\}$ which is colored by c_1.

The remaining colors c_2, c_3 and c_5 must be applied to the remaining three edges. Then by the same argument, c_3 (orange) must be applied to the edge $\{v_4, v_5\}$ as all other remaining edges are adjacent to $\{v_1, v_3\}$ which is colored by c_3.

Now there are two unused colors c_2 and c_5 and two uncolored edges. The color c_2 (red) can not be applied to $\{v_2, v_3\}$ forcing $\{v_3, v_4\}$ to be colored by c_2. This leaves $\{v_2, v_3\}$ to be colored by c_5 (green).

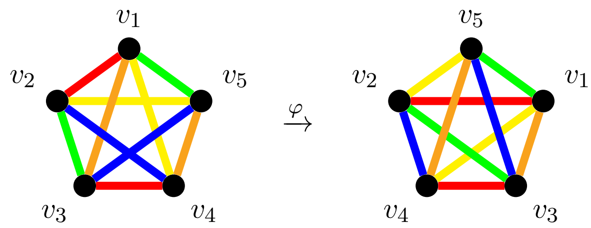


Figure 3.14: Applying graph isomorphism φ to the colored graph of Case 4C.

Define a graph isomorphism $\varphi(v_j) = v_{\sigma(j)}$ by $\sigma = (1,\ 5)(3,\ 4) \in S_5$. Then $G \cong G_{(5,5,2)}$ as seen in Figure 3.14.

Case 4.D: Assume $c(\{v_2, v_5\}) = c_4$ and $c(\{v_2, v_4\}) = c_3$.

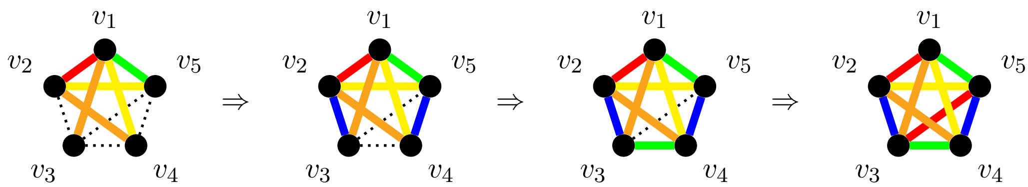


Figure 3.15: Illustrating the enumeration procedure as described in Case 4D.

Since c_3 and c_4 have been applied twice, the remaining colors are c_1, c_2 and c_5. These remaining three colors will be applied to the remaining four uncolored edges. Then since c_1 (blue) has not yet been applied, we must apply it to exactly two of the remaining edges. Among the remaining four edges, the only pair of non-adjacent edges is $\{v_2, v_3\}$ and $\{v_4, v_5\}$. Therefore both of these edges must be colored with c_1.

Now there are two remaining edges, $\{v_3, v_4\}$ and $\{v_3, v_5\}$, to be colored and two remaining colors to be used. Then $c(\{v_3, v_4\}) = c_5$, as $\{v_3, v_5\}$ is adjacent to an edge colored with c_5 (green). Finally the remaining edge $\{v_3, v_5\}$ must be colored with c_2 as shown in Figure 3.15.

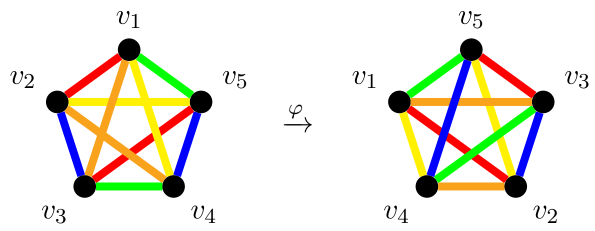


Figure 3.16: Applying graph isomorphism φ to the colored graph of Case 4D

In Figure 3.16, we see the final coloring. By applying the graph isomorphism $\varphi(v_j) = v_{\sigma(j)}$ where $\sigma = (1,\ 5,\ 3,\ 4,\ 2)$ to our final graph coloring we see G is clearly isomorphic to the graph $G_{(5,5,2)}$ in the Table 3.2.

Case 4.E: Assume $c(\{v_2, v_5\}) = c_4$ and $c(\{v_2, v_4\}) = c_5$.

Since c_4 and c_5 have both been applied twice, only the colors c_1, c_2 and c_3 may be applied to the remaining edges $\{v_2, v_3\}$, $\{v_3, v_4\}$, $\{v_3, v_5\}$ and $\{v_4, v_5\}$.

Again, since each color must be applied exactly twice, c_1 (blue) must be applied to exactly two edges; the only way to apply c_1 to exactly two of the remaining edges is by $c(\{v_2, v_3\}) = c_1$ and $c(\{v_4, v_5\}) = c_1$. Then the colors c_2 and c_3 must be applied to $\{v_3, v_4\}$ and $\{v_3, v_5\}$.

Both edges, $\{v_3, v_4\}$ and $\{v_3, v_5\}$, are adjacent to $\{v_1, v_3\}$, which is colored by c_3. Thus a contradiction arises as c_3 cannot be applied twice and the partial coloring cannot be completed.

Therefore it follows that if G is a uniform graph with 10 edges, 5 vertices, and 5 colors, G must be edge color isomorphic to $G_{(5,5,2)}$.

We see that each undirected uniform graph in Table 3.2 is non-isomorphic as an edge-colored graph to all undirected uniform graphs with the same number of vertices. Furthermore, since number of vertices is a graph isomorphism invariant, we find that each graph in our table is unique up to edge-colored graph isomorphism. $\square$

Chapter 4

Heisenberg-Like Lie Algebras

4.1 Heisenberg-Like Lie Algebras: Examples and Properties

4.1.1 Heisenberg-like Lie Algebras

In this chapter, we continue our discussion of two-step nilpotent Lie algebras by turning our attention to Heisenberg-like Lie algebras. Heisenberg-like Lie algebras were first introduced by Gornet and Mast as a generalization of Heisenberg-Type Lie algebras [10]. Throughout this chapter, we assume our arbitrary field is $\mathbb{R}$.

Definition 4.1.1. A *two-step nilpotent metric Lie algebra* $(\mathfrak{g}, \langle \cdot, \cdot \rangle)$ is a two-step nilpotent Lie algebra $\mathfrak{g}$ endowed with an inner product $\langle \cdot, \cdot \rangle$.

Heisenberg-type Lie algebras were first introduced by Kaplan in [14].

Definition 4.1.2 (From [5])**.** A two-step nilpotent metric Lie algebra $\mathfrak{g} = \mathfrak{v} \oplus \mathfrak{g}$ where $\mathfrak{g} = Z(\mathfrak{g})$ and $\mathfrak{v} = \mathfrak{z}^{\perp}$, is *of Heisenberg-type* if

$$J_z^2 = -|z| Id \,|_v$$

for all $z \in \mathfrak{z}$, where J_z is the Kaplan J_z map as defined in 2.1.16.

Let $\mathfrak{g}$ be a two-step nilpotent metric Lie algebra. We use the following notation adapted from [5]. Let $z \in Z$. Since J_z is skew-symmetric, we use the following notation:

- Denote the number of distinct eigenvalues of J_z^2 by μ_z and denote the μ_z distinct eigenvalues of J_z^2 by $\{-\theta_{1,z}^2, ..., -\theta_{\mu_z,z}^2\}$, where $0 \leq \theta_{1,z} \leq \theta_{2,z} \leq ... \leq \theta_{\mu_z,z}$.
- Then the distinct eigenvalues of J_z are $\{\pm i\theta_{1,z}, ..., \pm i\theta_{\mu_z,z}\}$.
- Let $W_m(z)$ be the invariant subspace of $\mathfrak{v}$ associated to $\theta_{m,z}$, $m = 1, ..., \mu_z$.

Heisenberg-like Lie algebras were defined in [10]. We use an equivalent definition found in [5].

Definition 4.1.3. A two-step nilpotent metric Lie algebra $(\mathfrak{g}, \langle\cdot,\cdot\rangle)$ is *Heisenberg-like* if

$$[J_z(x_m), x_m] \ \in \ \mathrm{span}_{\mathbb{R}}\{z\}$$

for all $z \in \mathfrak{z}$ and all $x_m \in W_m(z)$, $m = 1, ..., \mu$.

We use the following result from [5] to verify examples of Heisenberg-like Lie algebras.

Theorem 4.1.4. A two-step nilpotent Lie algebra with inner product $\langle\cdot,\cdot\rangle$ is Heisenberg-like if and only if for every $i = 1, ..., \mu$, there exists a constant $c_i \geq 0$ such that for every nonzero $z \in \mathfrak{z}$, $\theta_{i,z} = c_i|z|$.

Similar to uniform type Lie algebras, we may construct examples of Lie algebras from graphs.

Proposition 4.1.5. Let $\mathfrak{g}$ be a vector space with basis $\mathcal{B} = V \cup Z$. Let $G = (V, E)$ be a non-trivial simple directed graph with the further restriction if $(v_i, v_j) \in E$, $(v_j, v_i) \notin E$. Then $\mathfrak{g}$ is a Lie algebra with Lie bracket defined by

$$[v_i, v_j] = \begin{cases} c(v_i, v_j) & \text{if } (v_i, v_j) \in E \\ -c(v_j, v_i) & \text{if } (v_j, v_i) \in E \\ 0 & \text{otherwise} \end{cases}$$

and $[z_l, z_k] = [v_i, z_l] = [z_l, v_j] = 0$ for all $z_l, z_k \in Z$ and $v_i, v_j \in V$.

Proof. We omit this proof as it follows the proof of Theorem 3.2.10. $\square$

Remark. Let $\mathfrak{g}$ be a Lie algebra constructed from a graph $G = (V, E)$ with coloring function $c : E \to Z$ from the previous proposition. Endow $\mathfrak{g}$ with the inner-product $\langle \cdot, \cdot \rangle$ which makes $\mathcal{B} = V \cup Z$ orthonormal. Then $\mathfrak{g}$ is a metric Lie algebra with the following properties:

- The Lie algebra $\mathfrak{g}$ has a basis $\mathcal{B}$ such that $\mathcal{B} = V \cup Z$ where $V = \{v_1, \dots, v_q\}$ and $Z = \{z_1, \dots, z_p\}$ where $p, q > 1$ and $\dim(\mathfrak{g}) = p + q$.
- $\mathfrak{g}$ is a two-step nilpotent metric Lie algebra.
- Let $[v_i, v_j] \neq 0$ for some $v_i, v_j \in V$. Then if $[v_i, v_j] = \pm[v_i, v_k]$ for some $v_k \in V$, then $v_j = v_k$.
- If G has no isolated vertices (that is, for every $v_i \in V$ there exists an edge (v_i, v_j) or $(v_j, v_i) \in E$), then $Z(\mathfrak{g}) = \mathfrak{z}$.
- Let $z_l \in Z$. Then $J_{z_l} v_i = \pm v_j$ if and only if $[v_i, v_j] = \pm z_l$, where J_{z_l} is the linear map defined in 2.1.16.

If we endow the vector space $\mathfrak{g}$ with the inner product, $\langle \cdot, \cdot \rangle$, which makes $\mathcal{B} = V \cup Z$ orthonormal, then $\mathfrak{g}$ is a two-step nilpotent metric Lie algebra.

4.1.2 Examples

In this section we explore examples of known Heisenberg-like Lie algebras.

Example 4.1.6 (From [10])**.** Let $\mathfrak{g}$ be a vector space with basis $\mathcal{B} = \{v_1, v_2, v_3\} \cup \{z_1, z_2\}$. Endow $\mathfrak{g}$ with the inner product $\langle \cdot, \cdot \rangle$ which makes $\mathcal{B}$ orthonormal. Let $G = (V, E)$ be the directed edge-colored two-star graph in Figure 4.1 with edge coloring $c : E \to Z$ defined

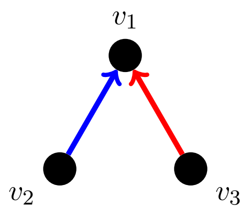


Figure 4.1: Edge colored two-star graph.

by $c(v_1, v_2) = z_1$ and $c(v_1, v_3) = z_2$. Then by Proposition 4.1.5 and the corresponding remark, $\mathfrak{g}$ is a two-step nilpotent metric Lie algebra. Furthermore, $\mathfrak{g}$ has Lie bracket $[\cdot, \cdot]$ defined in the proposition by

$$[v_1, v_2] = -[v_2, v_1] = z_1 \quad [v_1, v_3] = -[v_3, v_1] = z_2$$

and all other $[v_i, v_j]$, $[v_i, z_l]$ and $[z_l, z_k]$ null.

We now show $\mathfrak{g}$ is a Heisenberg-like Lie algebra. For $z_1, z_2 \in Z$, the maps J_{z_1} and J_{z_2} written in matrix form with respect to the basis $\{v_1, v_2, v_3\}$ of $\mathfrak{v}$ is

$$J_{z_1} = \begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \qquad \text{and} \qquad J_{z_2} = \begin{pmatrix} 0 & 0 & -1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}.$$

Let $z \in \mathfrak{z}$ and write $z = \alpha_1 z_1 + \alpha_2 z_2$. By linearity of the J map, $J_z = \alpha_1 J_{z_1} + \alpha_2 J_{z_2}$. Then J_z in matrix form is

$$J_z = \begin{pmatrix} 0 & -\alpha_1 & -\alpha_2 \\ \alpha_1 & 0 & 0 \\ \alpha_2 & 0 & 0 \end{pmatrix}$$

The characteristic polynomial of J_z is $p(t) = t(t^2 + |z|^2)$. It then follows J_z has eigenvalues $\{0, \pm i|z|\}$ where $|z| = \sqrt{\alpha_1^2 + \alpha_2^2}$. Therefore, $\mathfrak{g}$ is a Heisenberg-like Lie algebra.

Example 4.1.7 (From [10])**.** Let $\mathfrak{g}$ be a vector space with basis $\mathcal{B} = V \cup E$. Let $\langle \cdot, \cdot \rangle$ be the inner product which makes $\mathcal{B}$ orthonormal. Let $G = (V, E)$ be the edge-colored directed cycle graph in Figure 4.2 with coloring function $c : E \to Z$ defined by $c(v_1, v_2) = c(v_3, v_4) = z_1$ and $c(v_1, v_3) = v(v_2, v_4) = z_2$. Then by Proposition 4.1.5 $\mathfrak{g}$ is a two-step nilpotent metric Lie algebra. The Lie bracket $[\cdot, \cdot]$ of $\mathfrak{g}$ defined in the proposition is

$$[v_1, v_2] = [v_3, v_4] = z_1 \quad [v_1, v_3] = [v_2, v_4] = z_2.$$

We now demonstrate $\mathfrak{g}$ is a Heisenberg-like Lie algebra. The maps J_{z_1} and J_{z_2} are represented in matrix form with respect to the basis V are

$$J_{z_1} = \begin{pmatrix} 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \end{pmatrix} \quad \text{and} \quad J_{z_2} = \begin{pmatrix} 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}.$$

Let $z = \alpha_1 z_1 + \alpha_2 z_2 \in \mathfrak{z}$. Then $J_z = \alpha_1 J_{z_1} + \alpha_2 J_{z_2}$ and the map J_z written in matrix form is

$$J_z = \begin{pmatrix} 0 & -\alpha_1 & -\alpha_2 & 0 \\ \alpha_1 & 0 & 0 & -\alpha_2 \\ \alpha_2 & 0 & 0 & -\alpha_1 \\ 0 & \alpha_2 & \alpha_1 & 0 \end{pmatrix}.$$

The characteristic polynomial of J_z is $f(t) = (t^2 + |z|^2)^2$. We observe J_z has eigenvalues $\{0, \pm i|z|\}$ where $z = \sqrt{\alpha_1^2 + \alpha_2^2}$. Thus $\mathfrak{g}$ is a Heisenberg-like Lie algebra.

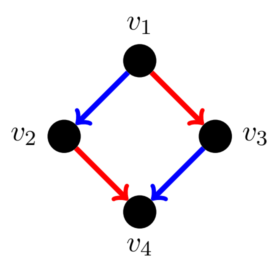


Figure 4.2: Edge-colored directed four cycle graph.

4.2 New Examples of Heisenberg-Like Lie Algebras

Here is our first new example. The following example was found independently by Mainkar [17].

Example 4.2.1. Let $\mathfrak{g}$ be a $(2n-1)$-dimensional vector space with basis $\mathcal{B} = V \cup Z$ where $V = \{v_1, \ldots, v_n\}$ and $Z = \{z_1, \ldots, z_{n-1}\}$. Endow $\mathfrak{g}$ with the inner product $\langle \cdot, \cdot \rangle$ which makes $\mathcal{B}$ orthonormal. Let $G = (V, E)$ be the directed n-star graph in Figure 4.3, where $E = \{(v_1, v_j) \mid 2 \leq j \leq n\}$. Define an edge coloring function $c : E \to Z$ by $c(e_1, e_j) = z_{j-1}$. By Proposition 4.1.5, $\mathfrak{g}$ is a Lie algebra with Lie bracket $[\cdot, \cdot]$ defined in the proposition.

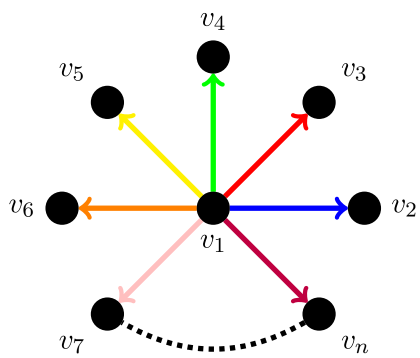


Figure 4.3: Edge-colored directed star graph.

We now show $\mathfrak{g}$ is a Heisenberg-like Lie algebra. Let $z = \sum_{l=1}^{n-1} \alpha_l z_l$. Then $J_z = \sum_{l=1}^{n-1} \alpha_l J_{z_l}$. In matrix form

$$J_z = \begin{pmatrix} 0 & \alpha_1 & \alpha_2 & \ldots & \alpha_{n-1} \\ -\alpha_1 & 0 & 0 & \ldots & 0 \\ -\alpha_2 & 0 & 0 & \ldots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ -\alpha_{n-1} & 0 & 0 & \ldots & 0 \end{pmatrix}$$

and has characteristic polynomial $p(t) = t^{n-2}(t^2 + |z|)$. Then J_z has eigenvalues $\{0, \pm i|z|\}$. Therefore $\mathfrak{g}$ is a Heisenberg-like Lie algebra.

Remark. In the previous example, each edge is uniquely colored. Thus, the direction of the edges may be switched and the corresponding Lie algebra $\mathfrak{g}$ will still be a Heisenberg-like Lie algebra.

Example 4.2.2. Let $\mathfrak{g}$ be the nine-dimensional vector space with basis $\mathcal{B} = V \cup Z$ where $V = \{v_1, \ldots, v_6\}$ and $Z = \{z_1, z_2, z_3\}$. Endow $\mathfrak{g}$ with the inner product $\langle \cdot, \cdot \rangle$ which makes $\mathcal{B}$ orthonormal. Let $G = (V, E)$ be the directed six-cycle in Figure 4.4, where $E = \{(v_i, v_{i+1}) \mid 1 \le i \le 5\} \cup \{(v_6, v_1)\}$. Define an edge coloring function $c : E \to Z$ by

$$c(v_1, v_2) = c(v_4, v_5) = z_1$$

$$c(v_2, v_3) = c(v_5, v_6) = z_2$$

$$c(v_3, v_4) = c(v_6, v_1) = z_3.$$

By Proposition 4.1.5, $\mathfrak{g}$ is a Lie algebra with Lie bracket $[\cdot, \cdot]$ defined in the proposition.

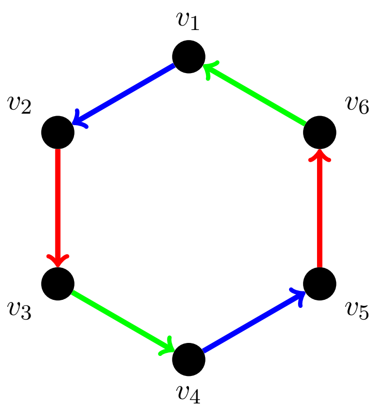


Figure 4.4: Edge-colored directed six cycle graph.

In particular, $\mathfrak{g}$ has bracket relations defined by

$$[v_1, v_2] = [v_4, v_5] = z_1, \quad [v_2, v_3] = [v_5, v_6] = z_2, \quad [v_3, v_4] = [v_6, v_1] = z_3.$$

We now show $\mathfrak{g}$ is a Heisenberg-like Lie algebra. Let $z = \alpha_1 z_1 + \alpha_2 z_2 + \alpha_3 z_3$. Then

$J_z = \alpha_1 J_{z_1} + \alpha_2 J_{z_2} + \alpha_3 J_{z_3}$. In matrix form, J_z, is given by

$$J_z = \begin{pmatrix} 0 & \alpha_1 & 0 & 0 & 0 & -\alpha_3 \\ -\alpha_1 & 0 & \alpha_2 & 0 & 0 & 0 \\ 0 & -\alpha_2 & 0 & \alpha_3 & 0 & 0 \\ 0 & 0 & -\alpha_3 & 0 & \alpha_1 & 0 \\ 0 & 0 & 0 & -\alpha_1 & 0 & \alpha_2 \\ \alpha_3 & 0 & 0 & 0 & -\alpha_2 & 0 \end{pmatrix}$$

and has characteristic polynomial $p(t) = t^2(t^2+|z|^2)^2$. Then J_z has eigenvalues $\{0, \pm i|z|\}$.

Therefore $\mathfrak{g}$ is a Heisenberg-like Lie algebra.

4.2.1 Graph Union

The following proposition details how two directed edge-colored graphs which give rise to two Heisenberg-like Lie algebras may be combined through graph union to create a new Heisenberg-like Lie algebra.

Proposition 4.2.3. Let $\mathfrak{g}_1$ be a vector space with basis $\mathcal{B}_1 = V_1 \cup Z$. Let $\langle \cdot, \cdot \rangle_1$ the inner product which makes $\mathcal{B}_1$ orthonormal. Let $G_1 = (V_1, E_1)$ be a directed edge-colored graph with color function $c_1 : E_1 \to Z$. Then G_1 along with c_1 define a Lie algebra on $\mathfrak{g}_1$.

Let $\mathfrak{g}_2$ be a vector space with basis $\mathcal{B}_2 = V_2 \cup Z$ where $V_1 \cap V_2 = \varnothing$. Let $\langle \cdot, \cdot \rangle$ be the inner product which makes $\mathcal{B}_2$ orthonormal. Let $G_2 = (V_2, E_2)$ be a directed edge-colored graph with color function $c_2 : E_2 \to Z_2$. Then G_2 along with c_2 define a Lie algebra on $\mathfrak{g}_2$. Suppose further $\mathfrak{g}_1$ and $\mathfrak{g}_2$ are Heisenberg-like Lie algebras.

Then the graph $G = (V_1 \cup V_2, E_1 \cup E_2)$ with color function $c : E_1 \cup E_2 \to Z$ defined by

$$c(e_i) = \begin{cases} c_1(e_i) & \text{if } e_i \in E_1 \\ c_2(e_i) & \text{if } e_i \in E_2 \end{cases}$$

defines a Lie algebra on $\mathfrak{g} = \mathfrak{v}_1 \oplus \mathfrak{v}_2 \oplus \mathfrak{z}$. Furthermore $\mathfrak{g}$ endowed with the inner product which makes $\mathcal{B}$ orthonormal is a Heisenberg-like Lie algebra.

Proof. By Proposition 4.1.5, $\mathfrak{g}_1$ and $\mathfrak{g}_2$ are Lie algebras with Lie bracket defined in the proposition.

Since $V_1 \cap V_2 = \varnothing$, $E_1 \cap E_2 = \varnothing$. Thus c is a well-defined surjective function. Then $\mathfrak{g} = \mathfrak{v}_1 \oplus \mathfrak{v}_2 \oplus \mathfrak{z}$ is a Lie algebra as defined in Proposition 4.1.5 with Lie bracket $[\cdot, \cdot]$ defined by the proposition.

Let $z \in \mathfrak{z}$. Since $E_1 \cap E_2 = \varnothing$, $[v_i, v_j] = 0$ for all $v_i \in V_1$ and $v_j \in V_2$. Then J_z

written in matrix form with respect to $V_1 \cup V_2$ is a block matrix and may be written

$$J_z = \begin{pmatrix} A & 0 \\ & \\ 0 & B \end{pmatrix}$$

where A is a $|V_1|$ by $|V_1|$ matrix and B is a $|V_2|$ by $|V_2|$ matrix.

Since $\mathfrak{g}_1$ and $\mathfrak{g}_2$ share the element z and are Heisenberg-like Lie algebras under their respective inner product, which makes Z orthonormal, and $\mathfrak{g}$ is an extension of both of these vector spaces, A must have eigenvalues $\{0, \pm i|z|\}$ and B must have eigenvalues $\{0, \pm i|z|\}$. Furthermore, since $\mathfrak{g}_1$ and $\mathfrak{g}_2$ are Heisenberg-like, the number of distinct eigenvalues is fixed regardless of $z \in \mathfrak{z}$. It follows $\mathfrak{g}$ is Heisenberg-like. $\square$

Chapter 5

Future Work

In this thesis, we explored uniform Lie algebras and their properties. There are many results yet unknown regarding these algebras. Specifically, in the partial classification for uniform graphs, we explored undirected graphs. It remains unknown how changing the directionality affects the underlying Lie algebra. Further research may include an investigation into the relationship between edge directionality and its effect on the uniform Lie algebra structure.

If $\mathfrak{g}$ is a uniform Lie algebra of type (p, q, r), it remains unknown if the value r is an invariant for all uniform bases of $\mathfrak{g}$. In particular, if r is shown to be an invariant, then new research may include a complete classification of uniform graphs by this invariant. Otherwise, if r is not an invariant, finding counterexamples to prove this may provide useful insight.

Finally, due to the limited number of examples of Heisenberg-like Lie algebras, further research constructing new infinite families of examples of these algebras has useful application. Such examples may be used to determine necessary and sufficient conditions in which Hesisenberg-like Lie algebras may be constructed by edge-colored graphs.

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