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CONNECTEDNESS OF TWO-SIDED CAYLEY DIGRAPHS

by

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Abstract

Cayley digraphs were introduced in 1878 by Arthur Cayley with the objective of understanding properties of groups. Two standard properties are that a Cayley digraph is connected if and only if the connection set generates the group and that all Cayley digraphs are vertex-transitive. As a result of this symmetry property, Cayley digraphs are used to model interconnection networks. Some generalizations of Cayley digraphs that relax the group axioms have been studied by Gauyacq, Kelarev and Praeger, and Mwambene. Another generalization of Cayley digraphs, called two-sided Cayley digraphs, was introduced in 2014 by Iradmusa and Praeger. As with Cayley digraphs, the group is the vertex-set, but two subsets are used to generate arcs. Iradmusa and Praeger determined sufficient conditions for a two-sided Cayley digraph to be connected. We generalize their result and give necessary and sufficient conditions for a two-sided Cayley digraph to be connected.

Chapter 1

Introduction

Cayley digraphs were introduced in 1878 by Arthur Cayley. The purpose was to visualize and better understand the properties of groups. Over the years Cayley digraphs have also been used to have equivalencies of expressions. Given a nonempty subset S of a group G, the Cayley digraph $\mathrm{Cay}(G, S)$ is the digraph with vertex set G such that (g, h) is an arc if and only if $hg^{-1} \in S$. The Cayley digraph has loops if and only if $e \in S$, and is undirected if and only if S is inverse closed. Since S is a set, and not a multiset, the digraph has no multiple arcs. Cayley graphs are used to model many interconnection networks because of their symmetry properties.

A number of generalizations of Cayley digraphs have been introduced, for example group action graphs by Annexstein et al. [2], Gauyacq's quasi-Cayley digraphs [6], semigroup graphs by Kelarev and Praeger [8], and groupoid graphs by Mwambene [11]. Whereas in [6], [8], and [11] the generalization is on the group structure where some axioms are relaxed, in [2] the elements of the connection set consist of permutations of vertices instead of group elements. In [7] Iradmusa and Praeger introduce a generalization of a Cayley digraph which they call a two-sided Cayley graph. Given nonempty subsets L and R of group G, the two-sided Cayley digraph $2\mathrm{SCay}(G; L, R)$ is the directed graph with vertex-set G and for each $g \in G$ and for each $l \in L$ and $r \in R$ there is an arc from g to $l^{-1}gr$. Unlike the Cayley digraph $\mathrm{Cay}(G, S)$ whose connection set is a subset S of group G, and like group action graphs, the connection set for $2\mathrm{SCay}(G; L, R)$ is a set of permutations. In this case the set of permutations is $\hat{S}(L, R) = \{\lambda_{l,r} \mid (l, r) \in L \times R\}$ such that for any $g \in G$ then $\lambda_{l,r}(g) = l^{-1}gr$.

Iradmusa and Praeger prove that for inverse-closed subsets L and R of G,

- the adjacency relation of $2\mathrm{SCay}(G; L, R)$ is symmetric if and only if $L^{-1}gR = LgR^{-1}$ for each $g \in G$;
- the graph has no loops if and only if $L^g \cap R = \emptyset$ for each $g \in G$; and
- the two-sided Cayley digraph $2\mathrm{SCay}(G; L, R)$ has no multiple arcs if and only if $(LL^{-1})^g \cap (RR^{-1}) = \{e\}$ for each $g \in G$.

The three properties $L^{-1}gR = LgR^{-1}$ for each $g \in G$, $L^g \cap R = \emptyset$ for each $g \in G$, and $(LL^{-1})^g \cap (RR^{-1}) = \{e\}$ for each $g \in G$ are collectively called the $2S$-Cayley property. In general, a two-sided Cayley graph is not a Cayley graph and it is not fully understood which two-sided Cayley graphs are isomorphic to one-sided Cayley graphs. However, Iradmusa and Praeger [7] give some sufficient conditions for a two-sided Cayley graph to be a Cayley graph. A special case is when the group G is abelian or if L or R is a subset of the center of G.

It is a standard result that a Cayley digraph $\mathrm{Cay}(G, S)$ is connected if and only if S generates the group G. To give conditions under which $2\mathrm{SCay}(G; L, R)$ is connected we first define a factorization of an element g of G by elements of L and R as follows. For nonempty subsets L and R of G, an L-R factorization of $g \in G$ is an expression $g = w_L w_R$ where w_L is a word in L and w_R is a word in R. Iradmusa and Praeger [7] prove that for nonempty inverse-closed subsets L and R with the $2S$-Cayley property, the digraph $2\mathrm{SCay}(G; L, R)$ is connected if and only if $G = \langle L \rangle \langle R \rangle$ and there exists an L-R factorization $ww' = e$ where $\ell(w)$ and $\ell(w')$ have opposite parity. If, given the same conditions on L and R, $G = \langle L \rangle \langle R \rangle$ but the second condition does not hold, it is proved in [7] that $2\mathrm{SCay}(G; L, R)$ is disconnected with exactly two components. In this dissertation we prove a modification of the above connectedness result. We will not require L and R to be inverse-closed nor to have the $2S$-Cayley property, but their elements will be required to have finite order. We show that the digraph is connected if and only if

1. $G = \langle L \rangle \langle R \rangle$ and
2. there exists an L^{-1}-R factorization $e = u_{L^{-1}} u_R$ of $e \in G$ such that $|\ell(u_{L^{-1}}) - \ell(u_R)| = 1$.

We give a direct proof, and for a finite group G we give an alternative proof that makes use of the fact that a finite directed graph is weakly connected if and only if it is strongly connected provided that at each vertex the in-degree and out-degree are equal. Another version of the connectedness theorem is stated, where L and R may have elements of infinite order, (L, R) need not have the $2S$-Cayley property, and in addition L and R are not assumed to be inverse-closed, as follows. Given nonempty subsets L and R of G, then $\mathrm{2SCay}(G; L, R)$ is connected if and only if

1. $G = \langle L^{-1} \rangle_{mon} \langle R \rangle_{mon}$ and $G = \langle L \rangle_{mon} \langle R^{-1} \rangle_{mon}$, and
2. there exist a $length + 1$ and a $length - 1$ L^{-1}-R factorization of e.

We further investigate disconnected two-sided Cayley digraphs. If condition 1 holds but condition 2 fails and either $L \cap L^{-1}$ or $R \cap R^{-1}$ is nonempty, we show that $\mathrm{2SCay}(G; L, R)$ is disconnected with exactly two components. We also prove that if both conditions fail, then the graph is disconnected with at least three components.

For any group G with $G \neq \langle L \rangle \langle R \rangle$, conditions under which $\mathrm{2SCay}(G; L, R)$ is disconnected with exactly two components have not been fully understood. Perhaps it will be fruitful to start with the special case where $G = I_2(n)$. For a dihedral group $I_2(n)$ of order $2n$, we prove that if L and R are subsets of $\langle r \rangle$ and if $r^i \in L$ and $r^j \in R$ with $\gcd(i - j, n) = 1$ and $\gcd(i + j, n) = 1$ then $\mathrm{2SCay}(I_2(n), L, R)$ is disconnected with exactly two components. We require more general subsets L and R that guarantee that $2\mathrm{SCay}(I_2(n), L, R)$ has exactly two components. It is observed that vertex- and edge-transitivity are important in applications. For non-empty subsets L and R of a group G [7] proves that if G factorizes as $G = N_G(L)N_G(R)$ then $2\mathrm{SCay}(G; L, R)$ is vertex-transitive. Finding necessary and sufficient conditions such that two-sided Cayley graphs are vertex-transitive or edge-transitive are still open questions.

In Chapter 2 some general graph theory is studied. This includes relevant definitions and results on connectedness and symmetry of graphs.

In Chapter 3 Cayley digraphs are studied, their properties, an application to interconnection networks and some generalizations of Cayley digraphs.

A new generalization of Cayley digraphs, called two-sided Cayley digraphs, is studied in Chapter 4. We provide a comparative study of the properties of one-sided and two-sided Cayley digraphs, focusing mainly on connectedness of the digraphs.

Chapter 2

Some Graph Theory

2.1 Preliminaries

This chapter treats some basic theory of general graphs mostly as given in [4] and [10]. Results on connectedness and vertex-transitivity of graphs are discussed in the second and third sections respectively. We begin with a definition of a graph and some examples.

Definition 2.1 A *graph*, Γ, is a pair of sets $V(\Gamma)$ and $E(\Gamma)$ where elements of $V(\Gamma)$ are called *vertices* and $E(\Gamma)$ is a set of unordered pairs of elements of $V(\Gamma)$ called *edges*. We call $V(\Gamma)$ and $E(\Gamma)$ the *vertex-set* and *edge-set* of Γ respectively. The number of vertices in a graph Γ is called the *order* of Γ. A graph of order 0 or 1 is called *trivial*.

When it is clear what graph we are referring to, we write V for $V(\Gamma)$ and E for $E(\Gamma)$. Unless otherwise stated we allow the vertex-set to be infinite, but require the graph to be locally finite, meaning that there are only finitely many edges containing any given vertex.

Example 2.2 Figure 2.1 shows a graph Γ on vertex-set $V = \{v_1, v_2, v_3, v_4, v_5, v_6, v_7\}$ with edge-set of size 6.

A more general concept than a graph is a directed graph or digraph for short.

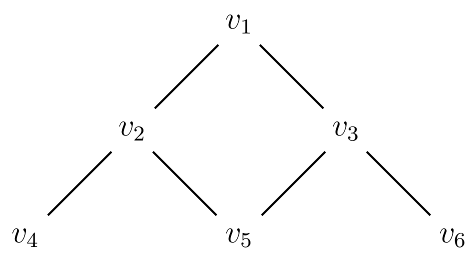

Figure 2.1: Graph

Definition 2.3 A *directed graph* or *digraph*, Γ, is a pair of sets $(V(\Gamma), A(\Gamma))$ where $A(\Gamma)$ consists of ordered pairs of distinct elements of $V(\Gamma)$ called *arcs*.

An arc (x, y) is said to go from x to y. An arc is represented by an arrow and an edge is represented by a line segment (with no arrow). In a digraph if both (x, y) and (y, x) are arcs, then the two are usually collapsed to become a single unordered edge. In a digraph Γ, if $(x, y) \in A$ if and only if $(y, x) \in A$, then Γ is in fact a graph.

Example 2.4 In Figure 2.2, the vertex-set is $V = \{v_1, v_2, v_3, v_4, v_5, v_6, v_7\}$, and $A = \{(v_1, v_2), (v_1, v_6), (v_2, v_3), (v_3, v_4), (v_4, v_5), (v_6, v_3), (v_6, v_5), (v_6, v_7)\}$ is the arc-set. Since for example $(v_1, v_2) \in A$ but $(v_2, v_1) \notin A$, then (v_1, v_2) is an arc and therefore $\Gamma = (V, A)$ is a digraph but not a graph.

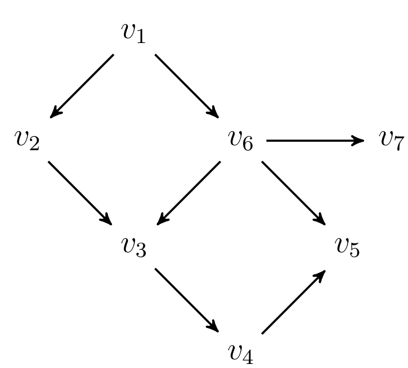

Figure 2.2: Directed graph

An edge that joins a vertex to itself is called a *loop*, and a graph is said to be *simple* if it has no loops and any two vertices are connected by at most one edge. Graphs with loops and/or multiple edges will be discussed only in Chapter 3.

An edge is written as an unordered pair $\{u, v\}$ or as uv whereas an arc is written as an ordered pair (u, v). An edge $\{u, v\}$ is said to *join* u and v or to be *incident* with vertices u and v. In this case vertices u and v are said to be *adjacent*, or to be *neighbors*. In a graph the number of vertices adjacent to a vertex v is called its *degree* and is denoted by $\deg(v)$. If in a graph all the vertices have the same degree, then the graph is said to be *regular*, or *k-regular* if the common degree is k. In a digraph, given vertex v, the *in-degree* of v is the number of arcs that go into v, and the *out-degree* is the number of arcs that go out of v. A digraph is *regular* if each vertex has the same in-degree and the same out-degree. A vertex with in-degree zero is called a *source* while a vertex with out-degree zero is called a *sink*.

Example 2.5 In the digraph in Figure 2.2 vertex v_6 has in-degree 1 since the only arc going into it is the arc from vertex v_1, and has out-degree 3 since arcs (v_6, v_3), (v_6, v_5) and (v_6, v_7) are the only arcs going out of v_6. Note that vertex v_1 in Figure 2.2 is a source and vertices v_5 and v_7 are sinks.

Definition 2.6 A *walk* from v_0 to v_n or a $v_0 - v_n$ *walk* is a sequence of vertices and arcs $v_0, e_1, v_1, ..., e_n, v_n$ in which each arc is $e_i = (v_{i-1}, v_i)$. A walk will be denoted by its sequence of vertices $v_0, v_1, ..., v_n$. The number n of arcs in a walk is called its *length*. A v_0-v_n walk is *closed* if $v_0 = v_n$. A walk in which no vertex is repeated is called a *path*, and a non-trivial closed path is called a *cycle*. A graph with no cycles is said to be *acyclic*.

Example 2.7 In the acyclic graph in Figure 2.2, v_1, v_6, v_3, v_4, v_5 is a path of length 4.

Given a graph Γ, if there is a path joining vertices u and v, the *distance* between the vertices u and v is the minimum length of a path from u to v. This distance is denoted by $d(u, v)$.

For a connected graph (see Definition 2.9), the distance function d is a metric on Γ since it satisfies the following properties for all vertices u, v and w;

- $0 \leq d(u, v) < \infty$,
- $d(u, v) = 0$ if and only if $u = v$,
- $d(u, v) = d(v, u)$,
- $d(u, v) \leq d(u, w) + d(w, v)$.

The *diameter* of Γ is the maximum distance between two distinct vertices in Γ and the *girth* is the length of the shortest cycle.

Example 2.8 In the complete graph 2.4 any two distinct vertices are adjacent. Hence, the diameter is 1. The shortest cycle in the complete graph has length 3. Hence the girth is 3. In Figure 2.1, the distance between vertices v_1 and v_2 in the graph is 1 while the distance between vertices v_4 and v_6 is 4. Since 4 is the maximum distance between two distinct vertices in the graph, the diameter of the graph is 4. The graph has a cycle v_1, v_2, v_5, v_3 of length 4 which happens to be the shortest and only cycle. Hence the girth of the graph is 4.

Suppose Γ_1 and Γ_2 are two graphs such that $V(\Gamma_1) \subseteq V(\Gamma_2)$ and $E(\Gamma_1) \subseteq E(\Gamma_2)$, then Γ_1 is a *subgraph* of Γ_2. The components of a graph that will be considered in the next section are examples of subgraphs.

Some common types of graphs are defined below.

A graph with n vertices and no edges is called an *empty graph* and is denoted by N_n. See Figure 2.3 for an empty graph on 6 vertices.

A graph with n vertices is called a *complete graph*, and is denoted K_n, if each of its vertices is adjacent to all the other vertices. In Figure 2.4 each vertex is adjacent to every other vertex. Hence the graph is complete.

A *cycle graph*, denoted C_n, has n vertices joined by n edges in an undirected cycle of length n.

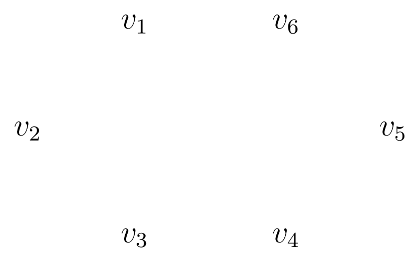

Figure 2.3: Empty graph N_6 on six vertices

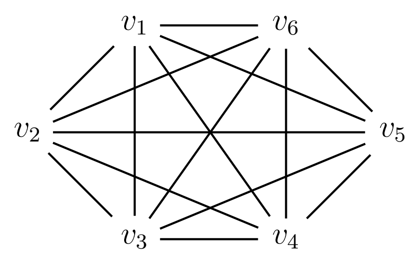

Figure 2.4: Complete graph K_6 on six vertices

A connected graph with no cycles is called a *tree*. A graph that does not have cycles is called a *forest*. Therefore a tree is a connected component of a forest. Observe that if we remove vertex v_5 in the graph of Figure 2.1 then the resulting graph is a tree and a forest.

An undirected path with n vertices and $n-1$ edges is called a *path graph* which is denoted by P_n. Observe that a path graph is a tree with two vertices of degree 1 and all other vertices of degree 2.

A graph Γ is *bipartite* if its vertex set can be partitioned into two nonempty subsets such that edges only join vertices that are in different subsets. If each vertex of one set in a bipartite graph is joined to every vertex of the other set and vice versa, then the bipartite graph is called a *complete bipartite graph* and is denoted $K_{m,n}$ if the two subsets have cardinality m and n. Figure 2.8 is a complete bipartite graph. A complete bipartite graph $K_{1,n}$ is called a *star graph*.

In Chapter 3 we will define Cayley graphs. It is a well-known fact that each Cayley graph is vertex-transitive (see Definition 2.25). However not all vertex-transitive graphs are Cayley graphs. The smallest vertex-transitive graph that

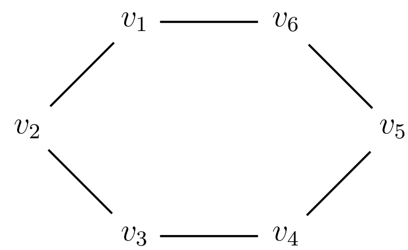

Figure 2.5: Cycle graph C_6 on six vertices

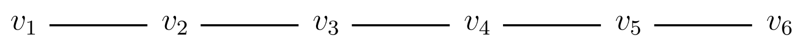

Figure 2.6: Path graph P_6 on six vertices

is not a Cayley graph is the *Petersen graph*. See Figure 2.9.

2.2 Connectedness of graphs

We consider connectedness of both graphs and digraphs. While connectedness is quite intuitive for graphs, the concept is slightly more subtle for directed graphs since strong connectedness takes direction into account while weak connectedness ignores direction. Some results on connectedness will also be explored, including some results that will be useful in the discussion of connectedness of two-sided Cayley graphs.

2.2.1 Connectedness of graphs

Definition 2.9 Two vertices g and h of a graph are said to be *connected* if $g = h$ or if $g \neq h$ and there is a path joining them. A graph is said to be *connected* if any two vertices are joined by a path in the graph. A graph that is not connected is said to be *disconnected*. A maximal connected subgraph of a graph is called a *connected component* or simply a *component* of the graph. We call vertex g of a graph Γ a *cut-vertex* if removing g partitions the component of the graph containing g into more than one component. If removing an edge e of a graph Γ partitions the component of the graph that contains e into more

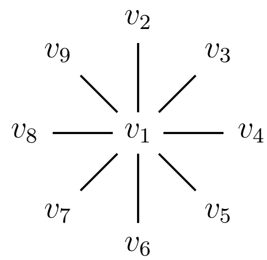

Figure 2.7: Star graph on 9 nodes

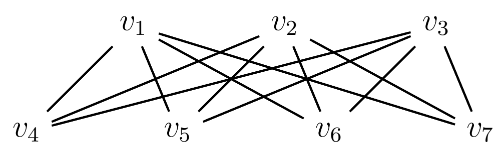

Figure 2.8: Complete bipartite graph $K_{3,4}$

than one component then we call e a *cut-edge*.

Example 2.10 The graph in Figure 2.11 is connected, but the graph in Figure 2.10 is disconnected since, for example, there is no path joining vertices v_1 and v_5. In Figure 2.10 removal of vertex v_6 increases the number of components of the graph. Hence v_6 is a cut-vertex. However there is no edge whose removal increases the number of components of the graph. Therefore the graph has no cut-edge. The graph in Figure 2.10 has two connected components $\{v_1, v_2, v_3\}$ and $\{v_4, v_5, v_6, v_7, v_8\}$.

If a graph is to model electrical power lines or cables of telephone network then it is not only important that the graph is connected, but that if one electrical power line develops a fault, there should be other line(s) through which power can be supplied to any consumer on the grid. To make this precise we give the following definition.

Definition 2.11 Let Γ be a graph. The *vertex-connectivity* or simply *connectivity* of the graph $\kappa(\Gamma)$ is the minimum number of vertices that must be removed to make Γ a disconnected graph or a trivial graph. Given $k \in \mathbb{N}$, a graph Γ is *k-connected* if there does not exist a set of k-1 vertices whose removal disconnects the graph.

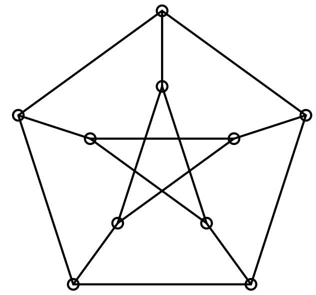

Figure 2.9: Petersen graph P

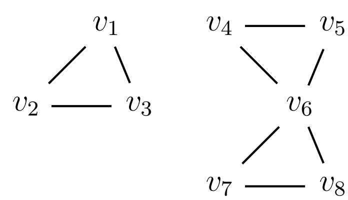

Figure 2.10: Cut vertices and cut edges

Note: Every graph is 0-connected and any connected graph is both 0- and 1-connected.

Example 2.12 Since any vertex of the path graph P_n of degree 2 is a cut-vertex, then $\kappa(P_n) = 1$. In a cycle there is no cut-vertex since a cycle is disconnected by removal of at least two nonadjacent vertices but not by removing only one vertex. Hence, $\kappa(C_n) = 2$. Since each pair of vertices of the complete graph on n vertices, K_n, is connected, the graph can only be disconnected by removing at least $n-1$ vertices (in fact becomes a trivial graph), meaning that $\kappa(K_n) = n - 1$. Since in a complete bipartite graph vertices of a partition set are adjacent to all the vertices of the other partition set, the graph will be disconnected at a minimum when at least all the vertices of the smaller partition set have been deleted. Therefore, $\kappa(K_{n,m}) = \min\{m, n\}$. Observe that by definition given any graph Γ if $\kappa(\Gamma) = n$, then the graph Γ is n-connected $(n-1)$-connected, $(n-2)$-connected, ..., 2-connected, 1-connected and 0-connected.

Example 2.13 The graph in Figure 2.11 is connected and cannot be disconnected by the removal of fewer than three vertices. Hence $\kappa(\Gamma) = 3$ and the graph is 3-connected, 2-connected, 1-connected, and 0-connected.

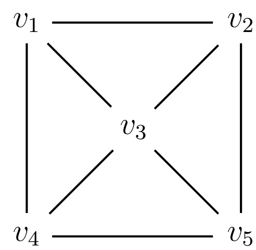

Figure 2.11: A 3-connected and 3-edge-connected graph

We also observe that in the graph of Figure 2.11 the removal of fewer than 3 edges will not disconnect the graph.

Definition 2.14 Given graph Γ, the *edge-connectivity* of the graph, $\lambda(\Gamma)$, is the minimum number of edges that if removed results in a disconnected graph or in a trivial graph. We say that a graph is *k-edge-connected* if there does not exist a set of $k-1$ edges whose removal disconnects the graph.

Example 2.15 If Γ is the graph in Figure 2.11, then $\lambda(\Gamma) = 3$.

Observe that if graph Γ_1 is obtained from the graph of Figure 2.11 by excluding edges $\{v_1, v_4\}$ and $\{v_2, v_5\}$, then $\kappa(\Gamma_1) = 1$ and $\lambda(\Gamma_1) = 2$. This illustrates that connectivity and edge-connectivity need not be equal.

Example 2.16 *Observe that* $\lambda(K_n) = n - 1$, $\lambda(C_n) = 2$, $\lambda(P_n) = 1$ *and* $\lambda(K_{m,n}) = \min\{m, n\}$.

The inequality $\lambda(\Gamma) \leq \delta(\Gamma)$, where $\delta(\Gamma)$ denotes the minimum degree of a vertex in Γ, is clear. Since removing a vertex also removes all the edges incident with it, connectivity based on edges is more stable than connectivity based on vertices, i.e. $\kappa(\Gamma) \leq \lambda(\Gamma)$. This means if a graph is k-edge connected then it is also k-vertex connected.

Proposition 2.2.1 *For any graph* Γ, $\kappa(\Gamma) \leq \lambda(\Gamma) \leq \delta(\Gamma)$. ∎

It is possible to characterize a 2-connected graph.

Lemma 2.2.2 *A graph Γ is 2-connected if and only if for any two vertices u and v of Γ there are two paths between them that are disjoint except at u and v.*

Proof Let Γ be a graph such that any two vertices u and v have two paths P_1 and P_2 joining them that are disjoint except at u and v. If one vertex from P_1, different from u and v, is removed then u and v remain connected by path P_2. Since u and v are arbitrary, Γ is 2-connected. Conversely suppose that Γ is 2-connected. Then Γ is connected. Hence for each pair of vertices u and v there is a path joining them. Suppose that there is a cut-vertex between u and v. This contradicts the assumption that Γ is 2-connected. Suppose that there are at least three paths joining u and v with no vertex common to all the paths except u and v. Then two disjoint paths paths between u and v can be constructed as follows. Starting at vertex u, follow the two outer paths. At each intersection follow the outer paths, until reaching vertex v. Therefore for any two vertices of the graph there are two disjoint paths joining them. ∎

Lemma 2.2.2 is a special case of Menger's Theorem [3] which we give below without proof. The proof of Menger's Theorem is by induction and is non-constructive (it does not give a systematic way to construct k vertex-disjoint paths for a given k-connected path).

Theorem 2.2.3 (Menger) *A graph Γ is k-connected if and only if every pair of vertices v and w has k paths that are pairwise disjoint except at u and v.*

Proposition 2.2.4 *For any pair of vertices of a 2-connected graph Γ, there is a cycle containing them.*

Proof Let Γ be a 2-connected graph and let u and v be vertices of Γ. Then, by Lemma 2.2.2, there are two disjoint paths joining u and v. These paths together with the vertices u and v form a cycle that contains the two vertices. ∎

In applications of graphs to computer networks edges represent links or connections. In a k-connected component if k-1 routers fail then the messages can still be routed within that component using the remaining routers.

Recall that a graph with no cycles is called a forest and a tree is a connected component of a forest. In a tree, any vertex of degree one is called a *leaf*.

Example 2.17 The graph in Figure 2.12 is a tree. Vertices v_1, v_4, v_6, v_8, v_{11}, v_{12}, v_{14} and v_{15} are leaves of the tree.

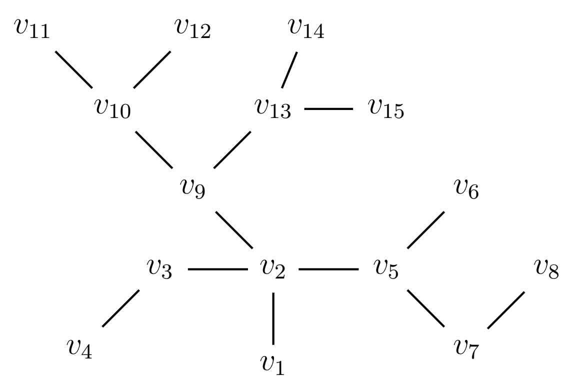

Figure 2.12: Tree

Proposition 2.2.5 characterizes a tree in different ways.

Proposition 2.2.5 *For any graph Γ the following statements are equivalent.*

(1) The graph Γ is a tree.

(2) For any two vertices of Γ there is a unique path in Γ connecting them.

(3) The graph Γ is minimally connected in the sense that Γ is connected but $\Gamma \setminus e$ is disconnected for every edge e of Γ.

(4) The graph Γ is maximally acyclic, that is Γ contains no cycle but $\Gamma \cup uv$ has a cycle if any two nonadjacent vertices $u, v \in V(\Gamma)$ are joined by an edge uv.

Proof Suppose that Γ is a tree. Let u and v be two distinct vertices of Γ with at least two paths in Γ joining them. If the paths only have u and v in common, then the two paths together with u and v give a cycle in Γ containing u and v, contradicting that Γ is a tree. If the paths intersect at least in one other vertex different from u and v, then they form a cycle that excludes both u and v if vertices adjacent to both u and v are common to both paths, otherwise the cycles contain either of the vertices. Each case contradicts that Γ is a tree. Hence there is only one path connecting u and v, proving that (1) implies (2).

Suppose that for any two vertices u and v of Γ there is a unique path joining them. If an edge e of the path is removed, then there is no path connecting vertices u and v. Therefore e is a cut-edge and hence $\Gamma \setminus e$ is disconnected. Therefore Γ is minimally connected, proving that (2) implies (3).

Suppose that Γ is minimally connected. If Γ had a cycle, then removing any edge e of the cycle would leave $\Gamma \setminus e$ disconnected. Hence Γ is acyclic. Let u and v be two nonadjacent vertices of Γ. Since Γ is connected, then there is a path p connecting u and v. Join vertices u and v to form edge uv. Hence $p \cup uv$ is a cycle in $\Gamma \cup uv$. But since u and v are arbitrary, $\Gamma \cup uv$ has a cycle if an edge uv is formed using any two nonadjacent vertices u and v of Γ, contradicting that Γ is minimally connected. Hence (3) implies (4).

Suppose that graph Γ is maximally acyclic. Then, by definition, Γ is a connected forest and hence a tree. Therefore (4) implies (1). ∎

Every nontrivial tree has at least two leaves, for example the ends of the longest path.

2.2.2 Connectedness of directed graphs

Connectedness for directed graphs could take directions into account. For that purpose strong connectedness is distinguished from weak connectedness and we say that a directed graph is connected if the digraph is strongly connected.

Definition 2.18 Two vertices u and v of a digraph are said to be *strongly connected* if there exists a directed u-v path and a directed v-u path in the digraph. We say that a digraph is *strongly connected* if for each ordered pair of vertices u and v of the digraph there is a directed path from u to v.

A digraph that is not strongly connected is said to be *disconnected.*

Example 2.19 In the digraph of Figure 2.13 there is a directed path from each vertex v_i to any other vertex v_j for $i, j \in \{1, 2, 3, 4, 5, 6\}$. Hence the digraph is strongly connected.

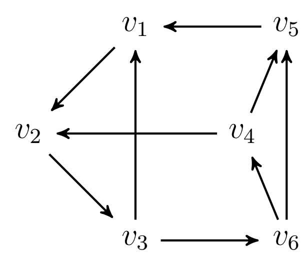

Figure 2.13: Strongly connected directed graph

For many digraphs, while the underlying graph may be connected, there may not be a directed path between some pairs of vertices.

Definition 2.20 In a digraph, vertex u is *weakly connected* to vertex v if there is a path $u = v_0, v_1, ..., v_n = v$ so that either (v_{i-1}, v_i) is an arc or (v_i, v_{i-1}) is an arc in the digraph. A digraph is *weakly connected* if any pair of its vertices is weakly connected, that is if the underlying graph is connected.

In particular, every strongly connected pair of vertices in a digraph is also weakly connected, but not vice versa.

Example 2.21 In the digraph of Figure 2.14, there is no directed path from vertex v_1 to vertex v_5. However, in the underlying graph there is a path from any vertex to any other vertex. Hence, the digraph is weakly connected.

We further consider the relation between strongly connected vertices.

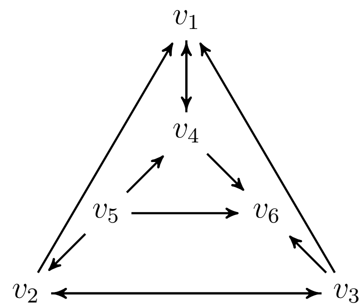

Figure 2.14: Weakly connected directed graph

Proposition 2.2.6 *Let* Γ *be a digraph and let a relation* $\sim$ *be defined on the vertex set of* Γ *by, for vertices* u *and* v*,* $u \sim v$ *if and only if* u *is strongly connected to* v*. This defines an equivalence relation on the vertex set* $V(\Gamma)$*.*

Proof Consider digraph Γ and let vertices u and v be related by $\sim$ if u is strongly connected to v. For each $u \in V(\Gamma)$, there is the trivial path of length zero from vertex u to itself. Hence $u \sim u$ and the relation $\sim$ is reflexive. Let u and v be in $V(\Gamma)$ such that $u \sim v$. Then there exist a u-v path and an v-u path in Γ, so we conclude that $v \sim u$ and $\sim$ is a symmetric relation. Suppose $u \sim v$ and $v \sim w$. Then there exist a u-v path, v-u path, v-w path, and w-v path in Γ. Hence, there exist a u-v-w path and a w-v-u path in Γ, implying that $u \sim w$, and hence $\sim$ is transitive. Therefore $\sim$ is an equivalence relation. ∎

The equivalence relation defined by strong connectedness partitions the vertex set $V(\Gamma)$ where the equivalence classes are the maximal strongly connected subdigraphs of the digraph, called *strongly connected components* or simply *strong components* of the digraph. Therefore, the equivalence class containing g will be written as Γ_g.

Observe that the directed graph in Figure 2.13 contains at least one directed cycle whereas the graph in Figure 2.14 does not have any directed cycle.

Definition 2.22 A digraph is called a *directed acyclic graph* if it contains no directed cycles.

The digraph given in Figure 2.14 is a directed acyclic graph.

Recall that given vertex v of a digraph, the number of arcs that go into v is called the in-degree of v, and the number of arcs that go out of v is called the out-degree of v. A vertex with in-degree zero is called a source while a vertex with out-degree zero is called a sink.

Note that Figure 2.14 has one sink and one source.

Proposition 2.2.7 *Every finite directed acyclic graph has at least one source and at least one sink.*

Proof Let $v_0v_1v_2...v_n$ be a longest path in a finite directed acyclic graph Γ. We claim that v_0 is a source and v_n is a sink.

Suppose, to the contrary, that there is an arc (u, v_0) in Γ. Then, either $u = v_i$ for some $i \in \{1, 2, ..., n\}$ or $u \notin \{v_1, ..., v_n\}$. If $u = v_i$ for some $i \in \{1, 2, ..., n\}$, then Γ contains a cycle $v_iv_0v_1...v_i$, which is a contradiction since Γ is acyclic. If $u \notin \{v_1, ..., v_n\}$, then $uv_0v_1...v_n$ is a path in Γ that is longer than the initial path. This contradiction proves that v_0 is a source.

A similar argument shows that v_n is a sink. ∎

Definition 2.23 Given a directed graph Γ, the *component graph* of Γ is the digraph formed by replacing each strong component by a single vertex, with an arc from component Γ_1 to component Γ_2 if there exists an arc (g, h) in Γ for some $g \in V(\Gamma_1)$ and some $h \in V(\Gamma_2)$.

Proposition 2.2.8 *For any digraph, the corresponding component graph is a directed acyclic graph.*

Proof Let Γ be a digraph and let $\Gamma_1, \Gamma_2, ..., \Gamma_n$ be the corresponding strong components of Γ. These components form the vertices of the component graph of Γ. Suppose, by re-labeling if necessary, that $\Gamma_1, \Gamma_2, ..., \Gamma_k, \Gamma_1$ is a directed cycle of the component graph. Then in the component graph any two vertices Γ_i, Γ_j with $1 \leq i \neq j \leq k$ are strongly connected. Since each Γ_i with $1 \leq i \leq k$ is strongly connected as a subgraph of Γ, then the union of $\Gamma_1, \Gamma_2, ..., \Gamma_k$ is a strongly connected subgraph of Γ. Hence $\Gamma_1, ..., \Gamma_k$ are in a single strongly connected component. This contradicts the assumption that $\Gamma_1, \Gamma_2, ..., \Gamma_n$ are maximal strong components of Γ. Therefore the component graph is acyclic. ∎

As noted above, every strongly connected graph is also weakly connected. However, the converse is not true. The graph in Figure 2.14 is weakly connected but not strongly connected since there is no directed path from vertex v_6 to vertex v_1 for example. The following proposition will be used in an indirect proof of Theorem 4.3.5 in case the graph there is finite.

Proposition 2.2.9 *Given a finite directed graph Γ, if at each vertex the in-degree and the out-degree are equal then Γ is weakly connected if and only if Γ is strongly connected.*

Proof Suppose Γ is a finite directed graph and at each vertex the in-degree equals the out-degree. If Γ is strongly connected, then Γ is weakly connected. Now, suppose Γ is weakly connected but not strongly connected. Consider the component graph of Γ. By Proposition 2.2.8 the component graph is a directed acyclic graph and hence by Proposition 2.2.7 it has at least one sink and at least one source. By hypothesis, in any strong component of Γ, the sum of the out-degrees equals the sum of the in-degrees. In particular, this is true for a sink Γ_1 of the component graph. Considering Γ_1 as a subgraph of Γ, the sum of its out-degrees equals the total number of arcs coming out of vertices in Γ_1 and going into vertices in Γ_1. Since Γ_1 is a sink of the component graph, there are arcs coming from other components to vertices in Γ_1 whereas all arcs coming out of Γ_1 end up in Γ_1. This contradicts the hypothesis that the sum of in-degrees and sum of out-degrees of Γ_1 are equal. Therefore, Γ is strongly connected. ∎

2.3 Symmetry of graphs

Consider a network of processors or communication ports. It is preferable for the network to look the same from any processor or communication port so that congestion is minimized and so that identical processors with the same routing algorithms may be used at each port. Graphs or digraphs are used to model interconnection networks where the ports are considered to be the vertices and communication channels are the edges or arcs. A graph or digraph with characteristics similar to those of the interconnection network just described is preferred.

The notion of isomorphism provides a way to compare any two given graphs or digraphs. In applications if two isomorphic (di)graphs model two situations, then the two situations being modeled should be similar in some way. Subsequently, vertex-transitivity and some of its consequences are discussed.

Definition 2.24 Let Γ_1 and Γ_2 be two digraphs. A bijection of vertex sets $\alpha : V(\Gamma_1) \to V(\Gamma_2)$ is called an *isomorphism* of graphs Γ_1 and Γ_2 if for vertices u and v of Γ_1, (u, v) is an arc in Γ_1 if and only if $(\alpha(u), \alpha(v))$ is an arc in Γ_2.

By definition every isomorphism preserves adjacency and nonadjacency of vertices. This implies that every isomorphism preserves the degree of a vertex. If vertex v has degree k, then so does its image under an isomorphism. Similarly, isomorphisms preserve the lengths of cycles that contain vertex v. If digraphs have multiple arcs an isomorphism must preserve the multiplicities as well.

A bijection from a graph to itself that preserves adjacency also preserves nonadjacency, that is is an isomorphism. An isomorphism of a graph with itself is called an *automorphism*. The set of automorphisms of a graph Γ under composition of functions is a group, denoted by $\mathrm{Aut}(\Gamma)$. To illustrate we give a list of graphs and their corresponding automorphism groups in Table 2.1, where S_n is the symmetric group of degree n, $I_2(n)$ is the dihedral group of order 2n, and C_n is the cyclic group of order n. Note that $S_m \times S_n$ is the direct product of groups S_m and S_n, and $S_m \wr C_2$ is the wreath product of S_m by C_2.

Table 2.1: Some automorphism groups of graphs up to isomorphism

Symbol	Graph	Automorphism group	Vertex-transitive	Edge-transitive
N_n	Empty	S_n	Yes	Yes
K_n	Complete	S_n	Yes	Yes
C_n	Cycle	$I_2(n)$	Yes	Yes
P_n	Path	$\mathbb{Z}_2$	Yes	Yes
$K_{m,n}$	Complete bipartite	$S_m \times S_n (m \neq n)$	Yes	Yes
		$S_m \wr \mathbb{Z}_2 (m = n)$	Yes	Yes
P	Petersen	S_5	Yes	Yes

For any graph there is a corresponding automorphism group. In [4] R. Frucht proved that given a group G, we can find a graph Γ for which G is isomorphic to its automorphism group, see Theorem 3.2.2.

The desire for every vertex in a network to look the same motivates the following definition.

Definition 2.25 A digraph Γ is *vertex-transitive* if for all $g, h \in V(\Gamma)$ there is an automorphism ϕ of the graph Γ such that $\phi(g) = h$.

In other words Γ is vertex-transitive if its automorphism group has a single orbit on the vertex set. In Proposition 3.2.1 we note that every Cayley digraph is vertex-transitive.

Example 2.26 The cyclic subgroup of order n in $\mathrm{Aut}(C_n) \cong I_2(n)$ acts vertex-transitively on the n-cycle C_n.

We now discuss some important consequences of the vertex-transitive property.

Proposition 2.3.1 *Every vertex-transitive digraph is regular.*

Proof Let u and v be distinct vertices of graph Γ. If $\phi \in \mathrm{Aut}(\Gamma)$ maps u to v, then it maps neighbors of u to neighbors of v in a one-to-one correspondence. ∎

Proposition 2.3.2 *If a graph is vertex-transitive and disconnected, then its connected components are all isomorphic.*

Proof Let Γ_1 and Γ_2 be connected components of a graph Γ. Since Γ is vertex-transitive, for each u in Γ_1 and for each v in Γ_2 there is an automorphism $\phi \in \mathrm{Aut}(\Gamma)$ such that $\phi(u) = v$. Consider ϕ_1 the restriction of ϕ on Γ_1. Note that ϕ_1 is a bijection between Γ_1 and $\phi_1(\Gamma_1)$. Observe that $\{u, v\}$ is an edge in Γ_1 if and only if $\{\phi_1(u), \phi_1(v)\}$ is an edge in Γ_2. Hence ϕ_1 is an isomorphism of the two subgraphs Γ_1 and Γ_2. ∎

Because of Proposition 2.3.2 it suffices to study connected vertex-transitive graphs.

Other forms of graph symmetry which are important in applications that involve interconnection networks include edge-transitivity, distance-transitivity and distance-regularity. However in this paper we will discuss only vertex- and edge-transitive properties and their importance in one application of Cayley graphs. A desirable property of an interconnection network is that if a path within the network develops a fault, there is still a path joining any two vertices of the network. This is called *fault-tolerance*. A property of Cayley graphs that ensures fault-tolerance is edge-transitivity which we define below.

Definition 2.27 A graph Γ is *edge-transitive* if, given any two edges $\{x, y\}$ and $\{u, v\}$, there exists an automorphism ϕ of Γ such that $\{\phi(x), \phi(y)\} = \{u, v\}$.

Example 2.28 Consider the star graph $K_{1,8}$ in Figure 2.7. If a map on $V(K_{1,8})$ sends vertex 1 to any other vertex, then adjacency of vertices will not be preserved. Hence the star graph is not vertex-transitive. However, given any two edges, a rotation through a given fixed angle transforms one edge into the other. Hence the star graph is edge-transitive.

For a digraph a concept analogous to edge-transitivity is arc-transitivity. A digraph Γ is *arc-transitive* if $\mathrm{Aut}(\Gamma)$ acts transitively on the arcs of Γ. Notice that an arc-transitive (di)graph is also edge-transitive and vertex-transitive. However, an edge-transitive graph may not be arc-transitive since edge-transitivity does not fix the order of the vertices. It is easy to prove that if a graph Γ has no isolated vertices and is edge-transitive but not vertex-transitive, then $\mathrm{Aut}(\Gamma)$ has exactly two orbits, and the orbits partition $V(\Gamma)$. In Theorem 3.2.8 we give necessary and sufficient conditions for a Cayley graph to be edge-transitive.

Chapter 3

Cayley Digraphs

3.1 Preliminaries

For any given group G and any nonempty subset S of G we consider a digraph with elements of G as vertices that displays the multiplicative structure of G relative to S, called a Cayley digraph. In subsequent sections we discuss some well-known results, an application to interconnection networks, and some generalizations of Cayley digraphs. We first define Cayley digraphs and give an example.

Definition 3.1 Let S be a nonempty subset of a group G. The *Cayley digraph* $\mathrm{Cay}(G, S)$ has vertex set G and for $g, h \in G$ a vertex pair (g, h) is an arc if and only if $hg^{-1} \in S$. The subset S is called the *connection set*.

This defines a left Cayley digraph. It is also possible to define a right Cayley digraph and if S^{-1} is used as its connection set then they are isomorphic. In Chapter 4 we will refer to a Cayley digraph as one-sided to distinguish it from the two-sided Cayley digraph defined in that chapter.

Example 3.2 Let $G = S_3$, and $S = \{(23), (123)\}$. Figure 3.1 shows the digraph $\mathrm{Cay}(G, S)$. Observe that successive multiplication of an element by (23) gives two opposite arcs between adjacent vertices, and hence give an edge. However, if we multiply an element by (123) there is no element in S that would give an opposite arc since $(123)^{-1} = (132) \notin S$. Hence the graph is directed. The graph has no loops, and we observe that the graph can only have a loop if e is in the connection set.

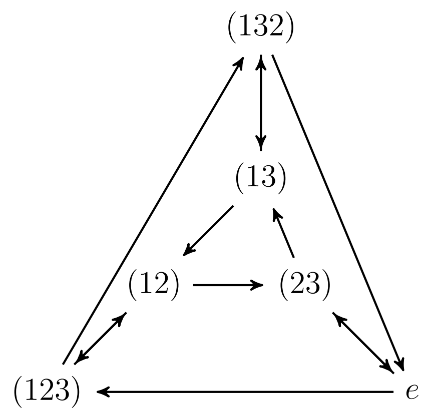

Figure 3.1: Directed Cayley digraph $\mathrm{Cay}(S_3, \{(23), (123)\})$

Remark 3.1.1 The adjacency relation for the Cayley digraph $\mathrm{Cay}(G, S)$ is symmetric (i.e., $\mathrm{Cay}(G, S)$ is an undirected graph) if and only if $S = S^{-1}$. A Cayley digraph has a loop at each vertex if and only if $e \in S$. For these reasons, many sources define a Cayley graph as requiring $S = S^{-1}$ and $e \notin S$, or restrict to such cases. If S is allowed to be a multiset rather than a set, then the Cayley digraph has multiple arcs. However in most cases, including throughout this chapter, S is just a set and the Cayley digraph does not have multiple arcs. The more general definition of two-sided Cayley digraphs given in Chapter 4 will allow multiple arcs as well as loops.

3.2 Some results on Cayley digraphs

Recall the following from Definition 2.24 and Definition 2.25. Two graphs are isomorphic if there is an adjacency and nonadjacency-preserving bijection between their vertex sets. If the two graphs are equal, then such a function is an automorphism. If for any two vertices of the graph there is an automorphism that maps one into the other, then the graph is vertex-transitive.

The following result is an important property which is useful in applications of Cayley digraphs.

Proposition 3.2.1 *Every Cayley digraph is vertex-transitive.*

Proof Let S be a nonempty subset of a group G and let $\Gamma = \mathrm{Cay}(G, S)$. For any $g \in G$, define a function $\rho_g : G \to G$, with $\rho_g(h) = hg$. Note that $\rho_g(h) = \rho_g(k)$ implies $hg = kg$, and hence $h = k$. If $k \in G$, then there exists $kg^{-1} \in G$ such that $\rho_g(kg^{-1}) = (kg^{-1})g = k$. Thus, ρ_g is a bijection on the vertex set of Γ. Let (h, sh) be an arc in Γ. Then

$$(\rho_g(h), \rho_g(sh)) = (hg, (sh)g) = (hg, s(hg)) = (\rho_g(h), s\rho_g(h)),$$

which is also an arc in Γ. Hence ρ_g is an automorphism of Γ (since ρ_g is an adjacency preserving bijection from a graph to itself) and therefore the permutation group $G_R = \{\rho_g \mid g \in G\}$ is a subgroup of $\mathrm{Aut}(\Gamma)$.

To show that G_R acts transitively on the vertices of the Cayley digraph, let h and k be vertices of Γ. Then equation $hg = k$ has solution $g = h^{-1}k$ in G and

$$\rho_g(h) = \rho_{h^{-1}k}(h) = h(h^{-1}k) = k.$$

Since h and k are arbitrary, G_R acts transitively on the vertices of $\mathrm{Cay}(G, S)$ and hence every Cayley digraph is vertex-transitive. ∎

Since every Cayley digraph $\mathrm{Cay}(G, S)$ is vertex-transitive, by Proposition 2.3.2 any disconnected digraph with components of different sizes cannot be a Cayley digraph. Similarly, by Proposition 2.3.1 any digraph that is not regular cannot be a Cayley digraph.

Suppose Γ is a vertex-transitive digraph. Then $\mathrm{Aut}(\Gamma)$ acts transitively on $V(\Gamma)$. Conversely, if G is a group then we can construct a digraph Γ such that G acts vertex-transitively on Γ as a group of automorphisms. The construction can be done as follows. Let $V(\Gamma) = G$ and let S be a nonempty subset of G and let $E(\Gamma)$ consist of all ordered pairs of the form (g, sg) where $g \in G$ and

$s \in S$. In this way we have constructed the Cayley digraph $\Gamma = \mathrm{Cay}(G, S)$. Let $S = \{s_1, s_2, \cdots, s_k\}$. In each arc (g, sg) insert a vertex and attach a directed path of length i to the new vertex. Then by Theorem 3.2 of [10], G is isomorphic to the automorphism group of this graph.

We have the following theorem.

Theorem 3.2.2 (Frucht) *Every finite group is isomorphic to the automorphism group of some digraph.*

Note that creating an undirected graph whose automorphisms are the same as for the constructed Cayley digraph requires a way to use edges to indicate and distinguish the various arc directions.

We define the following concept that will be helpful in this and subsequent sections.

Definition 3.3 Let S be a subset of a group G. A *word* in S is a string $w = s_1 s_2 \ldots s_k$ such that each $s_i \in S$.

We proved in Proposition 2.2.6 that for any digraph Γ the relation $\sim$ in $V(\Gamma)$ defined by

$$g \sim h \text{ if and only if } g \text{ is strongly connected to } h$$

is an equivalence relation and hence partitions $V(\Gamma)$. We observed that the class of $V(\Gamma)$ containing g is the strong component Γ_g containing g. We now describe the strong components of $\Gamma = \mathrm{Cay}(G, S)$.

Let G be a finite group and let $\Gamma = \mathrm{Cay}(G, S)$. For any $g \in V(\Gamma)$, if $h \in \Gamma_g$ then $h = sg$ for some word s in S and hence in $\langle S \rangle$. Therefore $h \in \langle S \rangle g$ and $\Gamma_g \subseteq \langle S \rangle g$. Now suppose $h \in \langle S \rangle g$. Then $h = w_s g$ where $w_s \in \langle S \rangle$. Since G is finite, every element of G has finite order. In particular, each $s \in S$ has finite order, and if the order of s is m then $s^{-1} = s^{m-1}$. Hence every word in $\langle S \rangle$ can be written as a word in S, and we conclude that there is a path from g to h. Therefore g and h are weakly connected. Observe that for each $s \in S$, every vertex g has a single arc with label s leading into g and a single arc labeled s leading out of g. Therefore since h and g are weakly connected, by Proposition 2.2.9, h and g are strongly connected, and hence there is also a directed path from h to g. Thus $h \sim g$ and we have $h \in \Gamma_g$ and hence $\langle S \rangle g \subseteq \Gamma_g$. Therefore, $\Gamma_g = \langle S \rangle g$.

As shown above, the components are the right cosets of the subgroup generated by S and are isomorphic as subgraphs. Observe that the number of components of Γ equals the number of right cosets of $\langle S \rangle$ which is given by the index $[G : \langle S \rangle]$. The case when $[G : \langle S \rangle] = 1$ is equivalent to $G = \langle S \rangle$. This proves the following proposition.

Proposition 3.2.3 *Let S be a nonempty subset of a finite group G, and let $\Gamma = \mathrm{Cay}(G, S)$. Then for each $g \in G$, $\Gamma_g = \langle S \rangle g$. Therefore, the number of components of Γ equals the number of right cosets, which is given by $[G : \langle S \rangle]$. In particular, $\mathrm{Cay}(G, S)$ is strongly connected if and only if $G = \langle S \rangle$.*

It follows that $\mathrm{Cay}(G, S)$ is disconnected with the number of components equal to the size of the group G if and only if $S = \{e\}$.

The following perspective will be useful for two-sided Cayley digraphs in Chapter 3. Consider a Cayley digraph $\mathrm{Cay}(G, S)$ and suppose that there is a path from e to g. Start at vertex e and move along the path from e to g. Write s_i if the path traverses an arc labeled s_i. Each time left multiply by the label of the arc traversed until vertex g is reached. In this way if the path traverses the sequence of edges $s_1, ..., s_n$ we write $s_n s_{n-1} ... s_2 s_1 e = g$ where $s_1, ..., s_n \in S$. This is a factorization of g making use of elements of S. Hence, the Cayley digraph $\mathrm{Cay}(G, S)$ is connected if and only if each element g of G can be written as a product of elements of S. Suppose that $|G| > 1$ and that the corresponding Cayley digraph is connected. Since each $g \in G$ has some factorization $g = s_n s_{n-1} ... s_2 s_1$, then

$$g^{-1} = s_1^{-1} s_2^{-1} ... s_{n-1}^{-1} s_n^{-1} = s'_m ... s'_2 s'_1$$

for some $s'_1, ..., s'_m \in S$. Therefore $e = gg^{-1} = s_n s_{n-1} ... s_2 s_1 s'_m ... s'_2 s'_1$ is a non-trivial factorization of e by elements of S.

Because it is desirable to work with Cayley digraphs due to their nice properties, it makes sense to ask when a graph is a Cayley digraph of a group. The answer to this question is given by Sabidussi's theorem.

Recall that given a group G and a nonempty set X, a group action of G on X is a map $G \times X \to X$ denoted by $g.x$ for $x \in X$ and $g \in G$, that satisfies the axioms:

1. $g_2.(g_1.x) = (g_2 g_1).x$ for all $x \in X$, and $g_1, g_2 \in G$ and
2. $e.x = x$ for all $x \in X$.

An action $G \times X \to X$ is said to be *free* if, for all $x \in X$, $g.x = x$ implies that $g = e$ (that is, only the identity fixes any element of X). A group with free action is said to *act freely.*

Theorem 3.2.4 (Sabidussi's Theorem) *A digraph* Γ *is a Cayley digraph of a group* G *if and only if it admits a free and transitive action of* G *on* $V(\Gamma)$.

Proof Suppose that G acts freely and transitively on Γ. Choose a vertex $u \in \Gamma$. Define $\phi : G \to V(\Gamma)$ by $\phi(g) = g(u)$ and $S = \{s \in G \mid (u, s(u)) \text{ is an arc in } \Gamma\}$. Now build $\mathrm{Cay}(G, S)$. We claim that ϕ is bijective. We note that each v in $V(\Gamma)$ is an image of u under some g by transitivity. Therefore ϕ is onto. If $g(u) = h(u)$, then $g^{-1}h(u) = u$. Because G acts freely, $g^{-1}h = e$, and $g = h$, implying that ϕ is one-to-one. Therefore ϕ is bijective.

We want to show that Γ is isomorphic to $\mathrm{Cay}(G, S)$. We observe that

$$(g, h) \text{ is an arc in } \mathrm{Cay}(G, S)$$

if and only if $hg^{-1} \in S$ if and only if each of the following is an arc in Γ

$$(u, hg^{-1}(u)),\ (u, (u.h).g^{-1}),\ (u.g, u.h),\ \text{and } (\phi(g), \phi(h)). \qquad \blacksquare$$

We observe that a similar result holds true for graphs rather than digraphs.

Remark 3.2.5

Let H be a subgroup of a finite group G. Identify subgroup H and all its left cosets as subgraphs of $\mathrm{Cay}(G,S)$. Take the cosets to be vertices and define two cosets aH and bH to be adjacent if and only if there exists $c \in aH$ and $d \in bH$ such that $c^{-1}d \in S$. Akers and Krishnamurthy [1] call such a graph a *quotient graph*. Lauri and Scapellato [10] call this a *coset graph* and denote it by $\mathrm{Cos}(G,H,S)$.

Praeger [3, page 186] defines a quotient graph in a more general way. Given a graph Γ and any partition $\mathcal{P}$ of its vertex set, the quotient graph has vertex-set $\mathcal{P}$ and an edge joining $P_1, P_2 \in \mathcal{P}$ if and only if there exist $v_1 \in P_1$ and $v_2 \in P_2$ that are adjacent in Γ. Notice that by this definition, a component graph and a coset graph are both quotient graphs.

When we refer to a coset graph in this subsection we mean as defined in [10] and will denote it by $\mathrm{Cos}(G,H,S)$.

Theorem 3.2.6 *Consider a Cayley digraph* $\mathrm{Cay}(G,S)$ *and let* H *be a subgroup of* G*. Then the coset graph* $\mathrm{Cos}(G,H,S)$ *is vertex-transitive.*

Before proving the theorem we observe that vertices aH and bH are adjacent if and only if the following equivalent statements hold:

$$\text{there exist } h_1, h_2 \in H \text{ such that } ah_1s = bh_2 \text{ for some } s \in S,$$

$$h_1 s = a^{-1}bh_2 \text{ for some } h_1, h_2 \in H \text{ and } s \in S,$$

$$a^{-1}b = h_1 s h_2^{-1} \text{ for some } h_1 s h_2^{-1} \in HSH, \text{ or}$$

$$a^{-1}b \in HSH.$$

We now prove the theorem.

Proof Let aH and bH be cosets of H in G. Then aH and bH are adjacent in $\mathrm{Cos}(G, H, S)$ if and only if $a^{-1}b \in HSH$ if and only if $(ga)^{-1}(gb) \in HSH$ if and only if gaH and gbH are adjacent in $\mathrm{Cos}(G, H, S)$ for all $g \in G$. Therefore, for all $g \in G$, λ_g defined by $\lambda_g(aH) = g(aH)$ is an endomorphism of $\mathrm{Cos}(G, H, S)$. The mapping λ_g is a bijection and hence an automorphism. Consider arbitrary vertices aH and bH of $\mathrm{Cos}(G, H, S)$. Then since $\lambda_{ba^{-1}}(aH) = bH$, $\mathrm{Cos}(G, H, S)$ is vertex-transitive. ∎

Vertex-transitive digraphs are characterized in the following result which is due to Sabidussi and is Theorem 3.8 of [10]. The proof uses the fact that all stabilizers in a transitive permutation group are conjugate.

Theorem 3.2.7 *Every vertex-transitive digraph is isomorphic to some coset graph* $\mathrm{Cos}(G, H, S)$.

Proof Let Γ be a vertex-transitive digraph and $G = \mathrm{Aut}(\Gamma)$. Consider vertex v of Γ and let $H = G_v = \{g \in G \mid gv = v\}$ be the stabilizer of v and $S = \{\sigma \in G \mid \sigma(v) \text{ is adjacent to } v\}$. Observe that $H \cap S = \emptyset$. Define $\phi : V(\mathrm{Cos}(G, H, S)) \to V(\Gamma)$ such that $\phi(\alpha H) = \alpha(v)$. Let $\alpha H = \beta H$. Then $\alpha^{-1}\beta \in H$ and by definition of H, $\alpha^{-1}\beta(v) = v$, implying that $\alpha(v) = \beta(v)$ and hence $\phi(\alpha H) = \phi(\beta H)$. Therefore ϕ is well-defined.

Now let $\phi(\alpha H) = \phi(\beta H)$. Then $\alpha(v) = \beta(v)$ and hence $\alpha^{-1}\beta(v) = v$, implying that $\alpha^{-1}\beta \in H$ and $\alpha H = \beta H$ and therefore ϕ is one-to-one. Take $v \in V(\Gamma)$. If there exists $g \in G$ such that $g(v) = v$, then $v = g(v) = \phi(H)$. If no such g exists in G, then v is adjacent to $\alpha(v) = \phi(\alpha H)$. Therefore $\phi(\alpha H) = v$ and therefore ϕ is onto. Hence ϕ is a bijection.

To show that ϕ is an isomorphism, let αH and βH be elements of $\mathrm{Cos}(G, H, S)$. We have αH and βH are adjacent in $\mathrm{Cos}(G, H, S)$ if and only if $\alpha^{-1}\beta \in HSH$ if and only if $\alpha^{-1}\beta = h_1 s h_2$ for some $h_1, h_2 \in H$ and $s \in S$ if and only if $h_1^{-1}\alpha^{-1}\beta h_2^{-1} = s$ if and only if $(v, h_1^{-1}\alpha^{-1}\beta h_2^{-1}(v)) \in E(\Gamma)$ if and only if $(v, \alpha^{-1}\beta(v)) \in E(\Gamma)$ if and only if $(\alpha(v), \beta(v)) \in E(\Gamma)$ if and only if $(\phi(\alpha H), \phi(\beta H)) \in E(\Gamma)$. Therefore ϕ is an isomorphism. ∎

An important property of undirected graphs in applications is that of edge-transitivity. For a Cayley graph $\text{Cay}(S_n, S)$ where $S \subset S_n$ with $|S| = d$, the following result was given in [1] as Theorem 2.

Theorem 3.2.8 *Let $S = \{s_1, \cdots, s_d\} \subset S_n$ and consider the Cayley graph $\text{Cay}(S_n, S)$. Then the Cayley graph is edge-transitive if and only if for each pair of generators s_1 and s_2 there is a permutation of the n symbols that maps the set of generators onto themselves, and in particular, maps s_1 into s_2.*

Proof Suppose that the Cayley graph is edge-transitive. Since each generator corresponds to an edge from the identity e, then by edge-transitivity for each pair of generators s_i and s_j there is an automorphism of the graph mapping s_i onto s_j.

Conversely, suppose that each pair of generators s_i and s_j can be mapped into each other by a permutation of the graph. Let $\{u, v\}$ and $\{x, y\}$ be two edges of the graph. Then there exists an automorphism $\rho_{u^{-1}x}$ such that $\rho_{u^{-1}x}(u) = x$. If the two edges correspond to the same generator s_i then $v = s_i u$ and $y = s_i x$. Therefore $\rho_{u^{-1}x}(v) = v(u^{-1}x) = s_i u(u^{-1}x) = s_i x = y$, and hence $\{u, v\}$ is mapped to $\{x, y\}$. Suppose that $\{u, v\}$ and $\{x, y\}$ correspond to two different generators. Then the automorphism $\rho_{u^{-1}x}$ maps $\{u, v\}$ to some edge $\{u(u^{-1}x), s_i u(u^{-1}x)\} = \{x, s_i x\}$. By hypothesis there exists an automorphism ψ that maps $\{x, s_i x\}$ to $\{x, y\}$. Hence $\{u, v\}$ is mapped to $\{x, y\}$ and therefore the Cayley graph is edge-transitive. $\blacksquare$

3.3 An application of Cayley digraphs to interconnection networks

We will discuss an application of Cayley digraphs to interconnection networks as first proposed in [1] and further studied in [9] and [12]. If we model interconnection networks by graphs, the vertices will correspond to processors, memory modules or switches, and the edges correspond to links. When the communication is one way, the graph is directed, and when the communication is two way the graph is undirected. The said papers discuss Cayley digraph properties that are useful for a good communication model. The properties include symmetry, connectivity, small diameter, small degree, and properties of specific Cayley digraphs as relates to the connection sets of the Cayley digraphs. All of [1], [9], and [12] suggest that the aim when designing interconnection networks is to have large networks that are symmetric, with small diameter and small degree and high connectivity, and offering simple routing algorithms.

Symmetry properties include vertex-transitivity, edge-transitivity, distance-transitivity and distance-regularity. In our study we will limit our discussion to the vertex-transitive properties and and only make short reference to edge-transitivity. The other symmetry properties are more complex for the Cayley graph case, and are expected to be quite complex for two-sided Cayley graphs.

The vertex-transitive property is important since a graph with the property looks the same from each vertex, thereby minimizing congestion since the load is distributed uniformly through all vertices. Edge-transitivity makes all the edges look the same and hence ensures that the load is distributed evenly along all links. By [9], an edge-transitive graph has more vertex-disjoint paths between any pair of vertices than any other and hence if a fault develops in a given link of the network, any two vertices of the graph will still be linked by another path. The diameter of a graph is also important because the smaller the diameter the smaller the delay in communication within a network since delay in communication is measured by the number of edges that a message has to pass through.

The following table, Table 3.2, adopted from [9], gives some common types of graphs used to model interconnection networks, together with a comparison of some of their symmetry properties, number of nodes, degree, and diameter.

In the following table $c(n)$ denotes the cycle $(12\cdots n)$ and for example $c^2(n) = (12\cdots n).(12\cdots n)$.

Table 3.1: Representation of some graphs as Cayley digraphs

Name of graph	Symbol	Cayley graph
Star	ST_n	$\mathrm{Cay}(S_n, \{(1i) \mid 2 \le i \le n\})$
Bubble-sort	BS_n	$\mathrm{Cay}(S_n, \{(ii+1) \mid 1 \le i \le n-1\})$
Modified bubble-sort	MB_n	$\mathrm{Cay}(S_n, \{(ii+1) \mid 1 \le i \le n-1\} \cup \{(1n)\})$
Binary hypercube	BC_n	$\mathrm{Cay}((\mathbb{Z}/2\mathbb{Z})^n, \{(1,0,\cdots,0),\cdots,(0,\cdots,0,1)\}$
Complete-transposition	CT_n	$\mathrm{Cay}(S_n, \{(ij) \mid 1 \le i < j \le n\})$
Alternating group	AG_n	$\mathrm{Cay}(A_n, \{(12i),(1i2) \mid 3 \le i \le n\})$
Complete	K_n	$\mathrm{Cay}(S_n, \{c^i(n) \mid 1 \le i \le n-1\})$

Table 3.2: Some Cayley digraphs and some of their characteristics

Cayley digraph	Vertex-transitive	Edge-transitive	Number of nodes	Degree	Diameter
ST_n	yes	yes	$n!$	$n-1$	$\lfloor\frac{3(n-1)}{2}\rfloor$
BS_n	yes	no	n!	$n-1$	$\frac{n(n-1)}{2}$
MB_n	yes	yes	$n!$	n	unknown
BC_n	yes	yes	2^n	n	n
CT_n	yes	yes	$n!$	$\frac{n(n-1)}{2}$	$n-1$
AG_n	yes	yes	$\frac{n!}{2}$	$2(n-2)$	$\lfloor\frac{3(n-1)}{2}\rfloor$
K_n	yes	yes	n	$n-1$	1

As can be seen in Table 3.2 the n-cube has 2^n vertices with degree n and diameter n whereas the star graph has $n!$ vertices with degree $n-1$ and diameter $\lfloor\frac{3(n-1)}{2}\rfloor$. For that reason the star graph gives a network with fewer edges and hence less communication delay than the n-cube. Hence the star graph is better in this regard. Observe that the connectivity of the n-cube is n and therefore a total of up to n-1 vertices may fail without disrupting the network. The star graph has both degree and connectivity equal to n-1 and therefore its fault tolerance is maximal.

Akers and Krishnamurthy, [1], introduced star and pancake graphs that they found to be superior to the n-dimensional binary hypercube and the cube-connected cycle networks considering their degree, diameter and connectivity. In [1] Cayley graphs were used to design symmetric interconnection networks. Such network could be decomposed recursively into smaller graphs with similar structure. This is significant since this allows a single model for many apparently different interconnection networks. Previously, each interconnection network was being modeled individually.

It was noted in [12] that a shortcoming of the star and pancake graphs is that there are not many of these graphs within a given range of the number of vertices since their vertex-sets grow factorially. A graph is said to be *hierarchical* if its generators can be ordered as $s_1, s_2, ..., s_d$ such that for all $1 < i \leq d$, s_i is not in the subgroup generated by the first $i-1$ generators. An importance of a hierarchical Cayley graph is that it has a recursive decomposition structure. Cayley graphs that are hierarchical under any ordering of the set of generators are called *strongly hierarchical*. An example of a strongly hierarchical Cayley graph is any *edge-transitive* Cayley graph. Exploiting the properties that for example an n-cube can be decomposed into two $(n-1)$-cubes interconnected by edges that are said to be in the n^{th} dimension, and similarly, an n-pancake graph can be seen as n copies of $(n-1)$-pancake graphs, bigger interconnection networks with the same structure can be built by duplicating an existing one.

A type of Cayley graph, $\text{Cay}(S_n, T)$ where T consists of transpositions of the form $(i, i+1)$, for $1 \leq i \leq n-1$, called a *bubble sort graph* was considered in [1]. The graph has $n!$ vertices, degree $n-1$, and diameter $\frac{n!}{2!(n-2)!} = \frac{n(n-1)}{2}$. Observe that in a bubble sort graph a path is a sequence of adjacent transpositions. Finding a route from a given permutation to the identity is done using the *bubble sort algorithm*. For example suppose we want to sort a sequence of numbers 3641257 using the bubble sort algorithm. Compare pairs of adjacent elements as follows. Passing from left to right, after comparing pairs, write the smaller one to the left and the bigger to the right. After moving through once, we have 3412567. The second time yields the sequence 3124567. A third pass yields 1234567.

Suppose that pancakes are labeled in order by the symbols $123...(i-1)i(i+1)...n$. Then flipping the first i pancakes yields $i(i-1)...321(i+1)(i+2)...n$. A pancake graph as a graph has vertices and generators representing pancake flips. Finding the diameter for a pancake graph is the same as finding the maximum number of flips needed to sort an arbitrary stack. The flipping algorithm is as follows. The first flip brings symbol n in first position. The second flip brings symbol n to n^{th} position. Now ignore n and make the next flip bring $n-1$ to the first position and the subsequent flip bring $n-1$ in the $(n-1)^{th}$ position, and so on. It was observed that the diameter and degree of the n-cube grow logarithmically as a function of the number of vertices, and the diameter and degree of the pancake graph grow at a rate lower than logarithmic.

In most cases fault-tolerance is one less than the connectivity and gives the largest number of vertices that can be removed without disconnecting the graph. Hierarchical Cayley graphs are maximally fault-tolerant, that is fault-tolerance is exactly one less than the degree.

Schibell and Stafford, [12], made use of the fact that a routing algorithm on a Cayley graph can be seen as some factoring in the corresponding group to produce a more efficient routing algorithm on Cayley graphs. Until their study, routing algorithms were designed for specific networks, and theirs was a first in designing a routing that works for a wide range of networks. In [12] a study of processor interconnection networks from Cayley graphs is done. The idea is to improve the speed and efficiency of supercomputers. The vertex-transitive property is seen to be important since at each processor the same routing algorithms may be used, and the symmetry of the graph reduces congestion. As observed in [1] the networks corresponding to the star and pancake graphs were found to be superior to the n-cube when considering degree, diameter, and connectivity. The paper makes reference to a number of results by Mckenzie(1984), Babai (1988) and Babai et al (1989) that give bounds on the diameter of Cayley graphs generated by permutations and Cayley graphs of nonabelian finite simple groups.

The routing problem was reduced to factoring of an element of a group G as a word in S, the generating set of $\mathrm{Cay}(G, S)$. For Cayley graphs of permutation groups with some special generators, a bound on the diameters of the corresponding graphs is given by making use of some special factorizations.

The paper by Lakshmivarahan, Jwo, and Dhall, [9], is a survey of symmetry properties of Cayley graphs of permutation groups. To determine how good a network is, the following are used: degree, diameter, node disjoint paths, optimal algorithms for various modes of packet communication, embeddability, symmetry properties, and recursive scalability. We note that most of the results on interconnection networks in [9] make reference to higher symmetries that we have not dealt with in our study. However the summaries given in Table 1, on the Cayley graphs and their generators; Table 4, comparison of symmetry among Cayley graphs; and Table 5, a comparison of topological properties of some Cayley graphs, are quite helpful.

3.4 Generalizations of Cayley digraphs

Cayley graphs have been generalized in the past in a few different ways. Examples of generalizations of Cayley graphs are Annexstein, Baumslag, and Rosenberg's group action graphs [2], Gauyacq's quasi-Cayley graphs [6], semigroup graphs by Kelarev and Praeger [8], and groupoid graphs by Mwambene [11]. In [2] what is generalized is the connection set, where instead of group elements the connection set consists of permutations of vertices of the graph. In [6], [8], and [11] the Cayley graph generalization is on the group structure where some axioms are relaxed. A brief overview of each of the generalizations and some of their properties is given below.

3.4.1 Group action graphs (GAGs)

Annexstein et al. introduce and study group action graphs as a generalization of Cayley graphs in [2]. Given a set of vertices V and a set of permutations Π

of set V, a *group action graph* (GAG) is defined to be the graph with vertex set V and for each $v \in V$ and each $\pi \in \Pi$ there is an arc from vertex v to vertex $\pi(v)$. An example of a GAG is a Cayley digraph in which each permutation is of the form λ_g with $\lambda_g(h) = gh$. Observe that a Cayley graph is a GAG (V, Π) where $V = \langle \Pi \rangle$ and for each $g \in V$ and $\pi \in \Pi$ there is an arc from g to $\pi.g$.

Annexstein et al. discuss a number of results that include:

1. Every weakly connected GAG is strongly connected. Hence, if (V, Π) is a connected GAG, then $\langle \Pi \rangle$ is a transitive group. i.e. for all $v_1, v_2 \in V$ there exists $\pi \in \langle \Pi \rangle$ such that $\pi(v_1) = v_2$. Recall: In a transitive permutation group, all stabilizers are conjugate. Hence we may refer to *the stabilizer subgroup* of a group G, denoted G_x for some $x \in G$.

2. Each connected group action graph (V, Π) is isomorphic to the coset graph $\mathrm{Cos}(\langle \Pi \rangle;\ G_x, \Pi)$.

   The following are some consequences of this theorem on the relation between a GAG (V, Π) and its induced Cayley graphs.

3. Each directed tree T which is a subgraph of GAG (V, Π) is a subgraph of $\mathrm{Cay}(\langle \Pi \rangle, \Pi)$ with multiplicity the size of $\langle \Pi \rangle_x$. It was noted that a cycle in a GAG may not result in a cycle in the induced Cayley graph since, while the initial and final arcs in the Cayley graph will start and end in the same coset, they may not be at the same vertex within the coset. However, 2 allows for a scheme that gives a point-to-point routing in a Cayley graph $(\langle \Pi \rangle, \Pi)$ from a similar scheme in an *associated GAG* (V, Π). The resulting routes are generally optimal and give an upper bound on diameter. Since $\mathrm{Cay}(\langle \Pi \rangle, \Pi)$ is symmetric, only optimal routings from the identity of $\mathrm{Cay}(\langle \Pi \rangle, \Pi)$ to any other vertex of the graph were considered. Notice that the shortest path from vertex u to vertex v follows the same path sequence of group generators as the shortest path from e to vu^{-1}.

4. **Corollary 3.4.1** *Let $F = (V, \Pi)$ be a connected GAG with associated*

Cayley graph $\Gamma = \mathrm{Cay}(\langle\Pi\rangle, \Pi)$*, and let* $H = \langle\Pi\rangle_x$*. Then*

$$\mathrm{Diam}(\Gamma) \leq \mathrm{Diam}_H(\Gamma) + \mathrm{Diam}(F).$$

If H *is generated by a subset* Ψ *of* Π*, then*

$$\mathrm{Diam}(\Gamma) \leq \mathrm{Diam}(\mathrm{Cay}(\langle\Psi\rangle, \Psi)) + \mathrm{Diam}(F).$$

An *edge-bisector* (vertex-bisector) for a given graph Γ is a set of edges (respectively, vertices) which if removed partitions Γ into two subgraphs with an equal number of vertices or differing by one.

5. If the GAG $\mathrm{Cos}(\langle\Pi\rangle; H, \Pi)$ has an edge-bisector (respectively, vertex-bisector) of size n, then $\mathrm{Cay}(\langle\Pi\rangle; \Pi)$ has an edge-bisector (respectively, vertex-bisector) of size $n.|H|$.

3.4.2 Quasi-Cayley digraphs

The generalization of Cayley digraphs to quasi-Cayley graphs discussed in [6] is a special case of a quasi-group digraph introduced by Dörfler (1974). A *groupoid* is a set with a binary operation. A groupoid Q is called a *quasi-group* if for each $a, b \in Q$ there exist unique $x, y \in Q$ such that $ax = b$ and $ya = b$.

Given a quasi-group Q and $S \subset Q$, the directed graph with vertex-set Q and with arc from q to qs for each $q \in Q$ and $s \in S$, denoted $QD(Q, S)$, is called a *quasi-group digraph*.

A subset S of a quasi-group Q is said to be *right-associative* if for every $a, b \in Q$, $(ab)S = a(bS)$. Consider a quasi-group Q with a right identity element e and S a right-associative subset of Q which generates Q and for any $s \in S$ if $sx = e$ then $x \in S$. A graph, denoted $QC(Q, S)$, with vertex-set Q and an edge joining q and qs for $q \in Q$ and $s \in S$ is called a *quasi-Cayley graph*. Dörfler gave the following analogue of Sabidussi's Theorem for quasi-group digraphs: Every digraph with a quasi-group representation involving a right-associative subset is a quasi-Cayley graph.

Thus, the group axioms are relaxed to form a groupoid, but the connection set S imposes some quasi-associativity, which is not associativity on elements. Some properties of quasi-Cayley graphs given in [6] are as follows.

1. Since the right identity element e is not in S, a quasi-Cayley digraph has no loops.

2. Since S generates Q, then $QC(Q, S)$ is connected.

3. Suppose that (g, h) is an arc of the quasi-Cayley graph. Then $h = gs$ for some $s \in S$. By hypothesis there exists $s_1 \in S$ such that $ss_1 = e$. Since S is right associative there exists $s_2 \in S$ such that $g = g(ss_1) = (gs)s_2 = hs_2$. We conclude that (h, g) is also a an arc and hence every quasi-Cayley graph is undirected.

4. Given vertices g and h of $QC(Q, S)$, since Q is a quasi-group there exists $q \in Q$ such that $gq = h$ and hence the quasi-Cayley graph is vertex-transitive.

5. The quasi-Cayley graph $QC(Q, S)$ is a Cayley graph if Q is a group.

In [6] a theorem analogous to Sabidussi's Theorem, Theorem 3.2.4, is given for quasi-Cayley graphs. To state the theorem, we need the following concept. For any graph $\Gamma = (V, E)$, a subset F of $\mathrm{Aut}(\Gamma)$ is called a *regular family* on V if for any vertices $u, v \in V$ there exists a unique automorphism $f \in F$ such that $f(u) = v$. Hence, if F is regular on V then $|F| = |V|$. The following generalization of Sabidussi's Theorem to quasi-Cayley graphs is Theorem 1 of [6].

Theorem 3.4.2 *For any connected graph $\Gamma = (V, E)$ the automorphism group of Γ contains a regular family on V if and only if Γ is a quasi-Cayley graph.*

If all paths from u to v are shortest paths then it is said that there is *routing of shortest paths.* A routing in which there is an equal number of paths through each vertex is called a *uniform routing.* In 1989 Heydmann, Meyer and Sotteau conjectured that in each vertex-transitive graph there is uniform routing of shortest paths. They proved that "in any quasi-Cayley graph there exists a uniform routing of shortest paths."

3.4.3 General semigroup graphs

In [8], Kelarev and Praeger define Cayley digraphs on semigroups. A semigroup is a set G with an associative binary operation. For any nonempty subset S of G $\mathrm{Cay}(G, S)$ is defined as in definition of right-sided Cayley graph. Since the existence of the identity element and the existence of an inverse for each element are not assumed, the existence and uniqueness of solutions of the equations $ax = b$ and $ya = b$ in G used in proving vertex-transitivity of $\mathrm{Cay}(G, S)$ is lost. Hence, $\mathrm{Cay}(G, S)$ is no longer guaranteed to be vertex-transitive when G is a semigroup. Kelarev and Praeger [8] give conditions for the recovery of the vertex-transitive property.

The automorphism group of $\mathrm{Cay}(G, S)$ is denoted in [8] by $\mathrm{Aut}_{\mathrm{S}}(G)$. Observe that each element of G acting by right multiplication defines an endomorphism of the Cayley digraph $\mathrm{Cay}(G, S)$.

To better understand results in [8] we define a number of concepts. The elements of S can be thought of as colors associated with the edges of the Cayley digraph. An endomorphism ϕ is color-preserving if for $x, y \in G$ and for all $s \in S$, if $sx = y$ then $s\phi(x) = \phi(y)$. The set of all color-preserving automorphisms is denoted by $\mathrm{ColAut}_{\mathrm{S}}(G)$). A semigroup G is said to be a *right zero band* if for all $x, y \in G$ the equation $xy = y$ holds.

The following concepts were also defined. These allow the conditions for vertex-transitivity of the Cayley digraph $\mathrm{Cay}(G, S)$ to be stated more clearly. A subset S of a semigroup G is called

1. a *subsemigroup* if SS is a subset of S,

2. a *left ideal* if GS is a subset of S (proper if $S \neq G$),

3. a *principal left ideal* if for some g_0 in G, $S = \{gg_0 \mid g \in G\}$.

Recall that a relation is a partial order if it is reflexive, antisymmetric, and transitive. A relation is a total or linear order if it is a partial order and any two elements are comparable. In addition, a semigroup is said to be *completely simple* if it has no proper ideals and has an idempotent minimal with respect to the partial order $e \leq f$ if and only if $e = ef = fe$.

We will only state the main results from the paper that give conditions needed for a Cayley digraph of a semigroup to be vertex-transitive. We state these without proof since the proofs are very involved.

Theorem 3.4.3 *Suppose S is a subset of a semigroup G such that all principal left ideals of the subsemigroup $\langle S \rangle$ are finite. Then the Cayley digraph* $\mathrm{Cay}(G, S)$ *is* $\mathrm{ColAut}_S(G)$*-vertex-transitive if and only if the following conditions hold:*

1. *$sG = G$ for all $s \in G$;*
2. *$\langle S \rangle$ is isomorphic to a direct product of a right zero band and a group;*
3. *$\mid \langle S \rangle g \mid$ is independent of the choice of $g \in G$.* ∎

Theorem 3.4.4 *Suppose S is a subset of a semigroup G such that all principal left ideals of the subsemigroup $\langle S \rangle$ are finite. Then the Cayley digraph* $\mathrm{Cay}(G, S)$ *is* $\mathrm{Aut}_S(G)$*-vertex-transitive if and only if the following conditions hold:*

1. *$SG = G$;*
2. *$\langle S \rangle$ is a completely simple semigroup;*
3. *the Cayley digraph* $\mathrm{Cay}(\langle S \rangle, S)$ *is* $\mathrm{Aut}_S(G)$*-vertex-transitive;*
4. *$\mid \langle S \rangle g \mid$ is independent of the choice of $g \in G$.* ∎

Lemma 5.1, Lemma 5.2, Corollary 5.3(ii) and Corollary 6.4(iii) of [8] give conditions under which the (strongly) connected component containing a given element equals the right coset of the subsemigroup $\langle S\rangle$ containing the same element, results that are analogous to Proposition 3.2.3.

Kelarev and Praeger also prove that under the hypothesis that G is a finite rectangular band, conditions 1 through 4 of Theorem 3.4.4 collapse to yield that $\mathrm{Cay}(G,S)$ is $\mathrm{Aut_S}(G)$-vertex-transitive if and only if $S\cap gG\neq\emptyset$ for all $g\in G$.

A final result is that if S is a subset of a semigroup G then the Cayley graph $\mathrm{Cay}(G,S)=\cup_{s\in S}\mathrm{Cay}(G,\{s\})$, and if $\mathrm{Cay}(G,S)$ is $\mathrm{ColAut}_S(G)$-vertex-transitive then $\mathrm{Cay}(G,\{s\})$ is $\mathrm{ColAut}_{\{s\}}(G)$-vertex-transitive for each $s\in S$. The paper ends by asking if the converse is true; whether $\mathrm{Cay}(G,\{s\})$ being $\mathrm{ColAut}_{\{s\}}(G)$-vertex-transitive for each $s\in S$ implies that all of $\mathrm{Cay}(G,S)$ is $\mathrm{ColAut}_S(G)$-vertex-transitive as well. This was proven false in general but true when G is a band or a completely simple semigroup by Jiang (*Semigroup Forum*, 2004).

3.4.4 Groupoid graphs

In [11] Mwambene considers graphs defined on groupoids. Given groupoid G, a *groupoid graph* is the digraph $\Gamma=(G,S)$ with vertex set G and edge set $\{(g,gs)|g\in G, s\in S\}$.

Mwambene refers to a subset S of a groupoid G as *quasi-associative* if for every $g,h\in G$, $g(hS)=(gh)S$ (i.e., right-associative in [6]). To generalize the conditions $e\notin S$ and $S=S^{-1}$ from the group setting, a subset S of a groupoid G is called a *Cayley subset* if for any $g\in G$ then $g\notin gS$ and for any $g\in G$ and $s\in S$, then $g\in g(sS)$. Observe that $g\notin gS$ implies that $g\neq gs$ for any $s\in S$. Hence, if S is a Cayley subset then the groupoid graph $\Gamma=\Gamma(G,S)$ has no loops. Observe that if $g\in g(sS)$ for all $g\in G$ and for all $s\in S$, then there exists $s_1\in S$ such that $g=g(ss_1)$. Notice that if S is also quasi-associative then $g=g(ss_1)=(gs)s_2$ for some $s_2\in S$ and hence in this case $\Gamma(G,S)$ is undirected.

For each $b \in G$, suppose that λ_b is the left translation defined by $\lambda_b(g) = bg$. If S is also quasi-associative, then $\lambda_b(gS) = (\lambda_b(g))S$. Now suppose that S is a quasi-associative Cayley subset. Consider an edge (g, gs). Then $\lambda_b(g, gs) = (bg, b(gs)) = (bg, (bg)s_1)$ is also an edge of Γ. Hence, λ_b is an endomorphism. Suppose that any left translation on G is an endomorphism of $\Gamma(G, S)$. Then if (g, gs) is an arc, then $(bg, b(gs))$ is an arc since λ_b is an endomorphism and hence has the form $(bg, (bg)s_1)$ for some s_1 in S. Hence, for any $g, b \in G$ and $s \in S$ there is an $s_1 \in S$ such that $b(gs) = (bg)s_1$. Therefore we have $b(gS) = (bg)S$ and hence S is quasi-associative. Hence S is quasi-associative if and only if every left translation on G is an endomorphism of $\Gamma(G, S)$.

As seen in 3.4.2 a groupoid G is a quasi-group if for any $a, b \in G$, there is a unique x such that $ax = b$ and a unique y such that $ya = b$. If in addition G has an identity element, then it is unique and G is called a *loop*. It was observed that Cayley digraphs described by quasi-associative Cayley subsets on quasi-groups are vertex-transitive. In fact all left multiplications are in $\mathrm{Aut}(\Gamma)$. We have already shown that every left multiplication is an endomorphism on the graph if S is quasi-associative. Now suppose that G is a quasi-group and g and h are vertices of the quasi-group graph. Then there is a unique $b \in G$ such that $h = bg = \lambda_b(g)$. Therefore, $\Gamma = \Gamma(G, S)$ is vertex-transitive.

It was observed that there is a generalization of Sabidussi's characterization of Cayley digraphs: A graph Γ is isomorphic to a Cayley digraph described by a quasi-associative Cayley subset on a quasi-group if and only if its automorphism group $\mathrm{Aut}(\Gamma)$ contains a subset that acts regularly on the vertex set $V(\Gamma)$. Such graphs are called *quasi-Cayley digraphs*. We have the following result: A graph is a Cayley digraph described by a quasi-associative Cayley subset on a left quasi-group with a right identity element if and only if its automorphism group $\mathrm{Aut}(\Gamma)$ acts transitively on its vertex set $V(\Gamma)$. It was also shown that a graph Γ is vertex-transitive if and only if Γ is isomorphic to a Cayley digraph $\Gamma(Q, S)$ such that S is quasi-associative on some left quasi-group Q with a right identity element.

Chapter 4

Two-Sided Cayley Digraphs

4.1 Preliminaries

In the last chapter Cayley digraphs, their properties and importance in applications were discussed as well as some generalizations. The generalizations to quasi-Cayley digraphs, semigroup graphs, and groupoid graphs entail relaxing the group axioms, and in group action graphs the generating set is a subset of permutations of the vertices of the graph as opposed to being a subset of the corresponding group. This chapter treats a generalization of Cayley digraphs due to Iradmusa and Praeger [7] called a two-sided Cayley digraph in which two subsets of a group are used to generate arcs. We begin with an overview of the work done in [7], focusing mainly on results that relate to connectedness of the two-sided Cayley digraphs. We then discuss some new results on connectedness of two-sided Cayley digraphs and in the final section we suggest possible areas of future inquiry.

Definition 4.1 Let L and R be nonempty subsets of a group G. The *two-sided Cayley digraph* $2\mathrm{SCay}(G; L, R)$ is a digraph with vertex set G and such that for each $g \in G$ and for each $l \in L$ and $r \in R$, there is an arc from g to $l^{-1}gr$. The *connection set* of $2\mathrm{SCay}(G; L, R)$ is the set of permutations $\hat{S}(L, R) = \{\lambda_{l,r} | (l, r) \in L \times R\}$, where for any $g \in G$, $\lambda_{l,r}(g) = l^{-1}gr$.

Example 4.2 Let $G = C_4 = \{e, a, a^2, a^3\}$ be the cyclic group of order 4 and let $L = \{a^2, a^3\}$ and $R = \{e, a, a^2, a^3\}$. Then $2\mathrm{SCay}(G; L, R)$ is a two-sided Cayley digraph with two edges between any two vertices and two loops at each vertex.

Remark 4.1.1 As was seen in Proposition 3.2.3 a Cayley graph $\mathrm{Cay}(G, S)$ is undirected if and only if S is inverse-closed, has loops if and only if $e \in S$, and never has multiple edges (unless S were to be a multiset). However, Example 4.2 illustrates that $2\mathrm{SCay}(G; L, R)$ may have multiple arcs even if L and R are not multisets, and can have loops even if $e \notin L$ and $e \notin R$.

For two-sided Cayley digraphs, Iradmusa and Praeger [7] define the $2S$-Cayley property which by Theorem 4.1.2 characterizes when a two-sided Cayley digraph is undirected, has no loops, and has no multiple edges, i.e., is undirected and simple.

Definition 4.3 Let G be a group with identity element e. If L and R are nonempty subsets of G, the pair (L, R) is said to have the *$2S$-Cayley Property* or be *$2S$-Cayley* if the following conditions hold.

1. $L^{-1}gR = LgR^{-1}$ for each $g \in G$;
2. $L^g \cap R = \emptyset$ for each $g \in G$; (where $L^g = g^{-1}Lg$)
3. $(LL^{-1})^g \cap (RR^{-1}) = \{e\}$ for each $g \in G$.

Theorem 4.1.2 *Let G be a group and let L and R be nonempty subsets of G. Then $\Gamma = 2\mathrm{SCay}(G; L, R)$ is a simple undirected graph if and only if (L, R) has the $2S$-Cayley property.*

Proof We prove the three parts of the $2S$-Cayley property are respectively equivalent to $\Gamma = 2\mathrm{SCay}(G; L, R)$ being undirected, having no loops and having no multiple arcs. The adjacency relation for any pair of vertices is symmetric if the number of opposite arcs between the vertices is equal.

The adjacency relation on Γ is symmetric if and only if any of the following equivalent statements hold.

$$(g,h) \text{ is an arc if and only if } (h,g) \text{ is an arc with equal multiplicities,}$$

$$h = l_i^{-1} g r_i \text{ for } l_i \in L,\ r_i \in R \text{ if and only if } g = l_j^{-1} h r_j \text{ for } l_j \in L,\ r_j \in R,$$
$$\text{with equal numbers of i's and j's}$$

$$h = l_1^{-1} g r_1 \text{ for } l_1 \in L,\ r_1 \in R \text{ if and only if } h = l_2 g r_2^{-1} \text{ for } l_2 \in L,\ r_2 \in R,$$
$$\text{with equal numbers of i's and j's}$$

$$h \in L^{-1} g R \text{ if and only if } h \in L g R^{-1} \text{ for each } g \in G \text{ with multiplicities, or}$$
$$L^{-1} g R = L g R^{-1} \text{ as multisets.}$$

The graph Γ has no loops is equivalent to the statement

$$\text{for each } g \in G \text{ and for all } l \in L \text{ and } r \in R, \text{ then } g \neq l^{-1} g r$$

which is equivalent to

$$\text{for each } g \in G,\ l \in L \text{ and } r \in R,\ r \neq g^{-1} l g,$$

which can be restated as

$$L^g \cap R = \emptyset \text{ for each } g \in G.$$

The graph Γ has no multiple arcs if and only if any of these three equivalent conditions holds.

If $l_1, l_2 \in L$ and $r_1, r_2 \in R$, then $l_1^{-1} g r_1 = l_2^{-1} g r_2$ iff $l_1 = l_2$ and $r_1 = r_2$ for all $g \in G$.

If $l_1, l_2 \in L$ and $r_1, r_2 \in R$, then $g^{-1} l_1 l_2^{-1} g = r_1 r_2^{-1}$ iff $g^{-1} l_1 l_2^{-1} g = r_1 r_2^{-1} = \{e\}$.

As sets,

$$(LL^{-1})^g \cap RR^{-1} = \{e\} \text{ for each } g \in G. \quad \blacksquare$$

Proposition 5.3 of [7] gives a simpler characterization of the $2S$-Cayley property when $G = N_G(L) N_G(R)$, which arises in trying to understand when $2\mathrm{SCay}(G; L, R)$ is vertex-transitive. If $G = N_G(L) N_G(R)$ then the conditions for (L, R) to have the $2S$-Cayley property become that $L^{-1} R = L R^{-1}$, $L \cap R = \emptyset$ and $L^{-1} L \cap R^{-1} R = \{e\}$.

4.2 Comparison between one- and two-sided Cayley digraphs

This section seeks to compare one-sided and two-sided Cayley digraphs. In particular we study some special circumstances under which a two-sided Cayley digraph is also a one-sided Cayley digraph. But first some isomorphisms of two-sided Cayley digraphs are discussed. If S is closed under inverses then the left and right Cayley digraphs on G with respect to S are isomorphic via the map $g \mapsto g^{-1}$. Similarly, if S is closed under conjugation, since $sg = g(g^{-1}sg)$ for all $s \in S$ and $g \in G$, then the left and right Cayley digraphs are isomorphic ($g \mapsto g$). The following result provides some isomorphisms amongst two-sided Cayley digraphs. It is given as Theorem 1.7 of [7] with the additional assumption that (L, R) is $2S$-Cayley, but also holds when the two-sided Cayley graph has loops, has multiple arcs, and/or is directed.

Theorem 4.2.1 *Let L and R be nonempty subsets of a group G, and let $x, y \in G$, and $\phi \in \mathrm{Aut}(G)$. Consider $\Gamma = \mathrm{2SCay}(G; L, R)$. Then*

$$\mathrm{2SCay}(G; L, R) \cong \mathrm{2SCay}(G; R, L) \cong \mathrm{2SCay}(G; L^\phi, R^\phi) \cong \mathrm{2SCay}(G; L^x, R^y).$$

Proof Let $\psi : G \to G$ be the bijection $g \mapsto g^{-1}$. It will be shown that ψ preserves adjacency and nonadjacency of vertices and also preserves multiple arcs. Let g be in $V(\Gamma)$, $l \in L$ and $r \in R$. Since $\psi^{-1} = \psi$, then $\psi(l^{-1}gr) = (l^{-1}gr)^{-1} = r^{-1}g^{-1}l = r^{-1}\psi(g)l$ and $\psi(\psi(g)) = g$ so $\psi(r^{-1}\psi(g)l) = l^{-1}gr$. Hence, $(g, l^{-1}gr)$ is an arc in $\mathrm{2SCay}(G; L, R)$ if and only if $(\psi(g), r^{-1}\psi(g)l)$ is an arc in $\mathrm{2SCay}(G; R, L)$. Observe also that for $l_1, l_2 \in L$ and $r_1, r_2 \in R$ and for $g \in G$, if $l_1^{-1}gr_1 = l_2^{-1}gr_2$ with $l_1 \neq l_2$ or $r_1 \neq r_2$ then applying ψ gives $r_1^{-1}g^{-1}l_1 = r_2^{-1}g^{-1}l_2$ with $l_1 \neq l_2$ or $r_1 \neq r_2$. Hence ψ preserves multiplicity of arcs incident with the same pair of vertices. Therefore, $\mathrm{2SCay}(G; L, R) \cong \mathrm{2SCay}(G; R, L)$.

Let $\phi \in \mathrm{Aut}(G)$ and denote $\phi : G \to G$ by $g \mapsto g^\phi$. By assumption ϕ is bijective. We have

$$(l^{-1}gr)^{\phi} = (l^{-1})^{\phi}(g)^{\phi}r^{\phi} = \phi(l)^{-1}\phi(g)\phi(r) = \phi(l)^{-1}\phi(g)\phi(r).$$

Similarly, $\phi^{-1}(\phi(l)^{-1}\psi_{\phi}(g)\phi(r)) = l^{-1}gr$. Hence $(g, l^{-1}gr)$ is an arc in the two-sided Cayley digraph $2\mathrm{SCay}(G; L, R)$ if and only if $(\psi_{\phi}(g), \phi(l)^{-1}\psi_{\phi}(g)\phi(r))$ is an arc in $2\mathrm{SCay}(G; L^{\phi}, R^{\phi})$. Notice that since ϕ is a well-defined one-to-one function, then by an argument similar to the one above, ψ_{ϕ} preserves multiple arcs. We conclude that $2\mathrm{SCay}(G; L, R) \cong 2\mathrm{SCay}(G; L^{\phi}, R^{\phi})$.

Let $x, y \in G$ and $\psi_{x,y} : G \to G$ with $\psi_{x,y} : g \mapsto x^{-1}gy$. Then $\psi_{x,y}$ is a bijection and

$$\psi_{x,y}(l^{-1}gr) = x^{-1}(l^{-1}gr)y = (l^{-1})^{x}(x^{-1}gy)r^{y} = (l^{x})^{-1}(x^{-1}gy)r^{y} = (l^{x})^{-1}\psi_{x,y}(g)r^{y}.$$

Observe also that $\psi_{x^{-1},y^{-1}} = \psi_{x,y}^{-1}$, and $\psi_{x^{-1},y^{-1}}((l^{x})-1\psi_{x,y}(g)r^{y}) = l^{-1}gr$. Hence, $(g, l^{-1}gr)$ is an arc in $2\mathrm{SCay}(G; L, R)$ if and only if $(\psi_{x,y}(g), (l^{x})^{-1}\psi_{x,y}(g)r^{y})$ is an arc in $2\mathrm{SCay}(G; L^{x}, R^{y})$. Observe that since conjugation is a bijection we have for example $x^{-1}l_1^{-1}x \neq x^{-1}l_2^{-1}x$ if and only if $l_1 \neq l_2$. Hence $\psi_{x,y}$ preserves multiple arcs. We conclude that $2\mathrm{SCay}(G; L, R) \cong 2\mathrm{SCay}(G; L^{x}, R^{y})$. ∎

Note that $2\mathrm{SCay}(G; \{e\}, R) = \mathrm{Cay}(G, R)$ where the graph $\mathrm{Cay}(G, R)$ is obtained by right multiplication, and $2\mathrm{SCay}(G; L, \{e\}) = \mathrm{Cay}(G, L^{-1})$ where the graph $\mathrm{Cay}(G, L^{-1})$ is obtained by left multiplication.

Now, consider the graph $2\mathrm{SCay}(G; L, R)$ where $G = C_5$ and $L = \{r\}$ and $R = \{r^4\}$. Then both $2\mathrm{SCay}(G; L, R)$ and $\mathrm{Cay}(G, \{r^3\})$ have the same digraph, shown in Figure 4.1, implying that the two-sided Cayley digraph $2\mathrm{SCay}(G; L, R)$ is a Cayley digraph. Also if either $L = \{e\}$ or $R = \{e\}$ then $\Gamma = 2\mathrm{SCay}(G; L, R)$ is a Cayley digraph. In Theorems 4.2.2 and 4.2.3 other instances in which a two-sided Cayley digraph is a Cayley digraph are given.

Theorem 4.2.2 *Let G be an abelian group, and let L and R be nonempty subsets of G with $LL^{-1} \cap RR^{-1} = \{e\}$. Then,* $2\mathrm{SCay}(G; L, R) = \mathrm{Cay}(G, L^{-1}R)$. *Therefore, for an abelian group, every two-sided Cayley graph is a Cayley graph.*

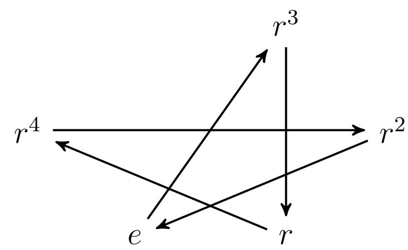

Figure 4.1: Graph of $2\mathrm{SCay}(C_5; \{r\}, \{r^4\})$ is the same as the graph of $\mathrm{Cay}(C_5, r^3)$

Proof Let G be an abelian group. Then the hypothesis $LL^{-1} \cap RR^{-1} = \{e\}$ is sufficient to guarantee $2\mathrm{SCay}(G; L, R)$ has no multiple edges and that $L^{-1}R$ is not a multiset. For $g, h \in G$, $h = l^{-1}gr$ for $l \in L$ and $r \in R$ if and only if $h = (l^{-1}r)g$ for $l^{-1}r \in L^{-1}R$ and hence (g, h) is an arc in $2\mathrm{SCay}(G; L, R)$ if and only if (g, h) is an arc in $\mathrm{Cay}(G, L^{-1}R)$. Therefore, $2\mathrm{SCay}(G; L, R) = \mathrm{Cay}(G; L^{-1}R)$. ∎

If connection sets of Cayley graphs are allowed to be multisets, then the hypothesis $LL^{-1} \cap RR^{-1} = \{e\}$ can be removed and the statement remains true for graphs with multiple edges. Since every Cayley graph is vertex-transitive, then for each abelian group G, the two-sided Cayley digraph $\Gamma = 2\mathrm{SCay}(G; L, R)$ is vertex-transitive and is connected if and only if $L^{-1}R$ generates G. For an abelian group G, if $L = \{g\} = R \subset G$ then since $L^{-1}R = \{e\}$, Γ is disconnected with the number of components equal to the size of G.

Let G_L and G_R defined by $G_L = \{\lambda_{g,e} \mid g \in G\}$ and $G_R = \{\lambda_{e,g} | g \in G\}$ be the regular permutation groups. Theorem 1.9 of [7], given below as Theorem 4.2.3, gives sufficient conditions for a two-sided Cayley digraph $2\mathrm{SCay}(G; L, R)$ to be a Cayley digraph with the assumption that (L, R) is $2S$-Cayley. The following result is a modification of Theorem 1.9 which removes most of the $2S$-Cayley hypothesis since we allow $\mathrm{Cay}(G, S)$ to be directed and have loops.

Theorem 4.2.3 *Let L and R be nonempty subsets of a group G such that $(LL^{-1})^g \cap (RR^{-1}) = \{e\}$ for all $g \in G$. Then the following hold.*

1. *$G_R \leq \mathrm{Aut}(\Gamma)$ if and only if $L^{-1}gR = L^{-1}Rg$ for each $g \in G$; here*

$\Gamma = \mathrm{Cay}(G; L^{-1}R)$.

2. *$G_L \leq \mathrm{Aut}(\Gamma)$ if and only if $L^{-1}gR = gL^{-1}R$ for each $g \in G$; here $\Gamma \cong \mathrm{Cay}(G; R^{-1}L)$*

Proof Let L, R be subsets of a group G and $(LL^{-1})^g \cap (RR^{-1}) = \{e\}$ for all $g \in G$. Therefore, as in the proof of Theorem 4.2.2, there are no multiple arcs in the Cayley digraph $\mathrm{Cay}(G, L^{-1}R)$. Suppose that $L^{-1}gR = L^{-1}Rg$. It will be shown that G_R is a subgroup of $\mathrm{Aut}(\Gamma)$. Given $x \in G$, $l \in L$ and $r \in R$, then $a = (x, l^{-1}xr)$ is an arc of Γ. Since $L^{-1}gR = L^{-1}Rg$ for all $g \in G$, then $a = (x, l^{-1}xr) = (x, (l')^{-1}r'x)$ for some $l' \in L$ and $r' \in R$. Take $\lambda_g \in G_R$. Then $\lambda_g(a) = (xg, l^{-1}xrg) = (xg, (l')^{-1}r'xg) = (xg, (l'')^{-1}xgr'')$ is an arc in Γ. Hence $\lambda_g \in \mathrm{Aut}(\Gamma)$ and thus $G_R \leq \mathrm{Aut}(\Gamma)$. Therefore, by Sabidussi's Theorem, Theorem 3.2.4, Γ is a Cayley digraph for group G. Each vertex x is adjacent to vertices in $L^{-1}Rx$, hence $\Gamma = \mathrm{Cay}(G; L^{-1}R)$.

Conversely, suppose that $G_R \leq \mathrm{Aut}(\Gamma)$. For $x \in G$, $l \in L$ and $r \in R$, then $l^{-1}xr \in L^{-1}xR$, it will be shown that $l^{-1}xr \in L^{-1}Rx$. Apply $\lambda_{x^{-1}} \in \mathrm{Aut}(\Gamma)$ to the arc $(x, l^{-1}xr)$ to get $(e, l^{-1}xrx^{-1})$. Hence $(e, l^{-1}xrx^{-1})$ is an arc in Γ. Therefore there exist $l_1 \in L$ and $r_1 \in R$ such that $l^{-1}xrx^{-1} = l_1^{-1}er_1$, that is $l^{-1}xr = l_1^{-1}r_1x$, and hence $l^{-1}xr \in L^{-1}Rx$. We therefore have $L^{-1}xR \subset L^{-1}Rx$. Let $l^{-1}rx \in L^{-1}Rx$. Applying $\lambda_{e,x}$ to the arc $(e, l^{-1}r)$ gives arc $(x, l^{-1}rx)$. But by definition an arc from x is of from $(x, l_1^{-1}xr_1)$. Hence $l^{-1}rx \in L^{-1}xR$ and therefore $L^{-1}Rx \subset L^{-1}xR$. This proves $L^{-1}xR = L^{-1}Rx$ for all $x \in G$, and hence $\Gamma = \mathrm{Cay}(G; L^{-1}R)$.

Similarly, $L^{-1}gR = gL^{-1}R$ for all $g \in G$ if and only if $G_L \leq \mathrm{Aut}(\Gamma)$, but this time $\Gamma \cong \mathrm{Cay}(G, R^{-1}L)$ with isomorphism $g \mapsto g^{-1}$. ∎

The example below illustrates some isomorphisms of graphs studied in Theorem 4.2.3 and in Theorem 4.2.1.

Example 4.4 1. Consider $G = S_3$, $L = \{(12), (13), (23)\}$ and $R = \{(123)\}$. Then $L^{-1}gR = gL^{-1}R$ for all $g \in S_3$. Therefore, by Theorem 4.2.3,

$G_L \leq \mathrm{Aut}(2\mathrm{SCay}(G; L, R))$ and hence $2\mathrm{SCay}(G; L, R) \cong \mathrm{Cay}(G; R^{-1}L)$, where $R^{-1}L = \{(12), (13), (23)\}$ and $G_L := \{\lambda_{g,e} | g \in G\}$.

2. For $L = \{(12), (13), (23)\}$, and $R = \{(123)\}$, by Theorem 4.2.1, for any permutation $\phi \in \mathrm{Aut}(S_3) \cong S_3$ and any elements $x, y \in S_3$,

   $2\mathrm{SCay}(G; L, R) \cong 2\mathrm{SCay}(G; R, L) \cong 2\mathrm{SCay}(G; L^\sigma, R^\sigma) \cong 2\mathrm{SCay}(G; L^x, R^y)$

3. Since the pair (L, R) in parts 1 and 2 is $2S$-Cayley, then by Theorem 4.2.1 the graphs of $2\mathrm{SCay}(G; R, L)$, $2\mathrm{SCay}(G; L^\sigma, R^\sigma)$, $2\mathrm{SCay}(G; L^x, R^y)$ are all connected because $\langle R^{-1}L \rangle = S_3$.

Theorem 4.2.3 includes the special case when the group G is abelian. Corollary 4.2.4 also gives some special circumstances when $2\mathrm{SCay}(G; L, R)$ is a Cayley digraph.

Corollary 4.2.4 *Let G be a group and let L and R be nonempty subsets of G with $(LL^{-1})^g \cap (RR^{-1}) = \{e\}$ for all $g \in G$ and let $\Gamma = 2\mathrm{SCay}(G; L, R)$. Further,*

1. *if $L \subset Z(G)$, then $\Gamma \cong \mathrm{Cay}(G, R^{-1}L)$.*
2. *if $R \subset Z(G)$, then $\Gamma = \mathrm{Cay}(G, L^{-1}R)$.*

Proof This follows by applying the above theorem since

1. if $L \subset Z(G)$, then for every $g \in G$, $L^{-1}gR = gL^{-1}R$, and
2. if $R \subset Z(G)$, then for every $g \in G$, $L^{-1}gR = L^{-1}Rg$. ∎

Proposition 5.1 of [7] gives other special circumstances under which a two-sided Cayley digraph is a Cayley digraph. Namely, given subsets L and R of a group G with the $2S$-Cayley property, then $\Gamma = 2\mathrm{SCay}(G; L, R)$ is a Cayley digraph if either $(N_G(L^{-1}) \cap N_G(L^{-1}R)).N_G(R) = G$, or $N_G(L^{-1}).(N_G(L^{-1}R) \cap N_G(R)) = G$. It is not understood more generally which $2\mathrm{SCay}(G; L, R)$ are Cayley digraphs. In fact Problem 3 of [7] asks if all $2\mathrm{SCay}(G; L, R)$ with $G = N_G(L)N_G(R)$ are Cayley graphs.

4.3 Connectedness of two-sided Cayley digraphs

This section mainly discusses results that relate to connectedness of two-sided Cayley digraphs. We start with a discussion of a result on connectedness of two-sided Cayley digraphs given in [7]. Our more general result is then given, together with some conditions under which different numbers of components may be obtained for some disconnected two-sided Cayley digraphs.

4.3.1 Connected two-sided Cayley graphs

Iradmusa and Praeger, [7], characterize when $\Gamma = 2\mathrm{SCay}(G; L, R)$ is connected assuming that the pair (L, R) is $2S$-Cayley and both L and R are inverse closed. In fact if L and R are inverse closed then Γ is undirected by Theorem 4.1.2 since the condition $L^{-1}gR = LgR^{-1}$ for all $g \in G$ in the definition of the 2S-Cayley property is trivially true in this case. Since connectedness is not affected by loops or multiples edges, the proof in [7] does not use anything further of the 2S-Cayley property. Theorem 1.5 of [7] is stated below, with the 2S-Cayley property omitted, as Theorem 4.3.1.

Before stating the theorem we give the following definitions of a factorization of elements of any group G and length of a word that will be used to characterize when any pair of group elements is connected.

Definition 4.5 Given a group G and nonempty subsets S and T of G, an S-T *factorization* of $g \in G$ is an expression $g = w_S w_T$ for g where w_S is a word in S and w_T is a word in T.

Definition 4.6 Let S be a subset of a group G. If $w = s_1 s_2 ... s_k$ is a word in S then the integer k is called the *length* of w, and is denoted $\ell(w)$.

Observe that if G is a group there can be many words equivalent to $g \in G$. The concept of length depends on the choice of expression and is not well-defined with respect to words equivalent in G.

Theorem 4.3.1 *Let L and R be inverse-closed subsets of a group G and $\Gamma = 2\mathrm{SCay}(G; L, R)$. Then Γ is connected if and only if*

1. *$G = \langle L\rangle\langle R\rangle$ and*
2. *there exists an L-R factorization $ww' = e$ of e where $\ell(w)$ and $\ell(w')$ have opposite parity.*

Proof Let L and R be inverse-closed subsets of a group G. Note that this implies the first part of the $2S$-Cayley property holds and hence $\Gamma = 2\mathrm{SCay}(G; L, R)$ is undirected by Proposition 4.1.2. It also implies that an L-R factorization of an element exists if and only if an L^{-1}-R factorization exists.

Suppose Γ is connected. Then every $g \in G$ can be written in the form

$$g = w_L e w_R = w_L w_R \tag{4.1}$$

for some L-R factorization $w_L w_R = g$ of g with $\ell(w_L) = \ell(w_R)$, proving that $G \subseteq \langle L\rangle\langle R\rangle$ and hence $G = \langle L\rangle\langle R\rangle$. Applying (4.1) to $g = r^{-1} \in R$ and rearranging yields

$$e = w_L w_R r = ww'$$

where $w = w_L$ and $w' = w_R r$ have lengths of opposite parity.

Conversely, suppose $G = \langle L\rangle\langle R\rangle$ and the second condition holds. Since $G = \langle L\rangle\langle R\rangle$, each $g \in G$ has an L-R factorization

$$g = w_L w_R = w_L e w_R. \tag{4.2}$$

If $\ell(w_L)$ and $\ell(w_R)$ have opposite parity then in (4.2) replace e with ww' where w and w' are words in L and R respectively such that $\ell(w)$ and $\ell(w')$ have opposite parity. As a result, we may assume that for each $g \in G$ there is an L-R factorization w_L, w_R with lengths $\ell(w_L)$ and $\ell(w_R)$ of the same parity.

If $\ell(w_L) = \ell(w_R)$ then the resulting expression $g = w_L w_R = w_L e w_R$ shows there is a path from e to g. If $\ell(w_L) \neq \ell(w_R)$, then note $e = ll^{-1} = rr^{-1}$ since L and R are nonempty, and thus $e \in L^2 \cap R^2$ since L and R are inverse-closed. Inserting $\frac{|\ell(w_L)-\ell(w_R)|}{2}$ copies of e written in the appropriate form into whichever word is shorter will again demonstrate that there is a path from e to g. Since Γ is undirected, in both cases there is also a path from g to e and this proves Γ is connected. ∎

A consequence of Proposition 3.2.1 is that if a Cayley digraph is disconnected, then all its components are isomorphic. This result shows that the following two-sided Cayley graph is not a Cayley graph.

Example 4.7 As in [7, Example 2.3], let $G = I_2(6) = \langle r, s | r^6 = s^2 = e, rsrs = e\rangle$ and let $L = \{rs, r^3, e\}$ and $R = \{s\}$. Then the simple undirected graph $2\mathrm{SCay}(G; L, R)$ is disconnected since $\langle L\rangle\langle R\rangle = \{e, s, r, rs, r^3, r^3s, r^4, r^4s\} \neq G$, (see Theorem 4.3.1). The two-sided Cayley graph has the following two non-isomorphic components $\{e, s, r, rs, r^3, r^4, r^3s, r^4s\}$ and $\{r^2, r^5, r^2s, r^5s\}$. See Figure 4.2. Hence by Proposition 2.3.2 and Proposition 3.2.1 the graph is not a Cayley digraph.

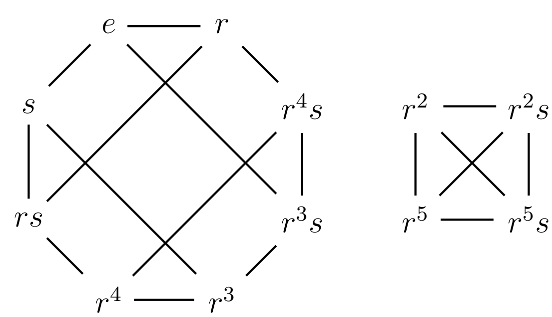

Figure 4.2: Two-sided Cayley digraph with non-isomorphic components

In the following example some hypotheses of Theorem 4.3.1 do not hold, but the two-sided Cayley digraph is connected.

Example 4.8 Consider group $G = S_3$ and let $L = \{(12), (13), (23)\}$ and $R = \{(123)\}$. Then $\langle L\rangle = S_3$ and $\langle R\rangle = \{e, (123), (132)\}$. Hence $\langle L\rangle\langle R\rangle = G$. Since $L = \{(12), (13), (23)\} = L^{-1}$ and $R = \{(123)\} \neq \{(132)\} = R^{-1}$, then L is inverse closed but R is not. Note that $w = (12)(13) = (132)$ is a word in L and $w' = (123)$ is a word in R with lengths of opposite parity and $ww' = (132)(123) = e$. Despite R not being inverse-closed, the two-sided

Cayley digraph $\Gamma = 2\text{SCay}(G; L, R) \cong K_{3,3}$ is connected, as seen in Figure 4.3.

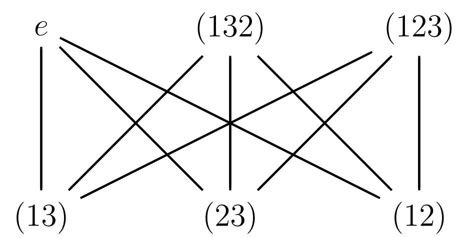

Figure 4.3: Graph of $2\text{SCay}(S_3; \{(12), (13), (23)\}, \{(123)\})$

Example 4.8 illustrates that there are two-sided Cayley digraphs $2\text{SCay}(G; L, R)$ that are connected but for which L or R is not inverse-closed or the pair (L, R) does not satisfy the $2S$-Cayley property. Our main result, Theorem 4.3.5, generalizes Theorem 1.5 from [7], included here as Theorem 4.3.1, to characterize when any two-sided Cayley digraph is connected and thereby explains Example 4.8. We begin with some lemmas useful for manipulating factorizations of elements of G.

We allow the possibility that G is an infinite group and use nonempty subsets of G that consist of elements of finite order. But note that if G is finite, the results will apply to any nonempty subsets of G. If a result applies only for finite groups or finite graphs we will state that.

Lemma 4.3.2 *Let G be a group and let S be a nonempty subset consisting of elements of finite order. For any word w_S in S there is a word $w_{S^{-1}}$ in S^{-1} so that $w_S = w_{S^{-1}}$ as elements in G.*

Proof Let w_S be a word in S. For any letter $s \in S$ appearing in w_S that is of order m in G, $s = (s^{-1})^{m-1}$. Replacing all letters in w_S in this way yields a word $w_{S^{-1}}$ in S^{-1} so that $w_S = w_{S^{-1}}$. ∎

Lemma 4.3.3 *Let G be a group and let L and R be nonempty subsets of G that consist of elements of finite order. If there exists an $\langle L\rangle$-$\langle R\rangle$ factorization* *of $g \in G$ then there exists an L^{-1}-R factorization and an L-R^{-1} factorization of g.*

Proof Let L and R be nonempty subsets of G that consist of elements of finite order. Let $g = w_{\langle L \rangle} w_{\langle R \rangle}$ be an $\langle L \rangle$-$\langle R \rangle$ factorization of $g \in G$. In particular, $w_{\langle L \rangle}$ and $w_{\langle R \rangle}$ are words in $L \cup L^{-1}$ and $R \cup R^{-1}$ respectively. Applying Lemma 4.3.2 to any letter $l \in L$ appearing in $w_{\langle L \rangle}$ and to any letter $r^{-1} \in R^{-1}$ appearing in $w_{\langle R \rangle}$ yields an L^{-1}-R factorization. Applying Lemma 4.3.2 to any letter $l^{-1} \in L^{-1}$ appearing in $w_{\langle L \rangle}$ and to any letter $r \in R$ appearing in $w_{\langle R \rangle}$ yields an L-R^{-1} factorization. $\blacksquare$

Although we will not use this observation, note that a similar argument proves that there is also an L-R and an L^{-1}-R^{-1} factorization of $g \in G$. Since all four such factorizations are in particular $\langle L \rangle$-$\langle R \rangle$ factorizations, in fact any of the five factorizations can be converted to any other.

Definition 4.9 Let S and T be nonempty subsets of a group G. Given an integer k, an S-T factorization $g = w_S w_T$ of $g \in G$ will be said to satisfy the *length* $+\, k$ *condition* or be a *length* $+\, k$ *S-T factorization* of g if $\ell(w_T) = \ell(w_S) + k$.

The cases $k = 1$ or $k = -1$ will be particularly relevant and we combine these and say $g = w_S w_T$ satisfies the *length*± 1 *condition* if $|\ell(w_T) - \ell(w_S)| = 1$. The following lemma provides an important relationship between the length$+1$ and length-1 conditions and will be used to adjust relative lengths of factorizations in the proofs of Theorem 4.3.5 and Proposition 4.3.8.

Lemma 4.3.4 *Let G be a group and let S and T be nonempty subsets of G that consist of elements of finite order. There exists a length $+\,1$ S-T factorization of the identity element $e \in G$ if and only if there is a length $-\,1$ S-T factorization of e and in either case there is a length$+k$ S-T factorization of e for every integer k.*

Proof Let S and T be nonempty subsets of a group G that consist of elements of finite order. Suppose there exists an S-T factorization $e = w_S w_T$ with $\ell(w_T) = \ell(w_S) + 1$.

Using that S consists of elements of finite order, choose $s \in S$ of order m. If $m = 1$, then $s = e$ and $e = (e^2 w_S)w_T$ is an S-T factorization of e with $\ell(w_T) = \ell(e^2 w_S) - 1$. If $m > 1$, then $e = (s^m w_S)w_T$ and successively inserting $e = w_S w_T$ between the S word and T word $m - 2$ times in the factorization yields the S-T factorization $e = (s^m w_S^{m-1})w_T^{m-1}$ where

$$\ell(w_T^{m-1}) = (m-1)(\ell(w_S)+1) = (m-1)\ell(w_S) + m - 1 = \ell(s^m w_S^{m-1}) - 1.$$

Similarly, if $e = w_S w_T$ with $\ell(w_T) = \ell(w_S) - 1$ then it is possible to use $t \in T$ of finite order to produce a length $+1$ S-T factorization of e.

Lastly, if $e = w_S w_T$ is a length $+1$ S-T factorization and $e = x_S x_T$ is a length -1 S-T factorization, then note that

$e = x_S^k x_T^k$ is a length $-k$ S-T factorization with $k > 0$,

$e = w_S x_S x_T w_T$ is an equal length S-T factorization, and

$e = w_S^k w_T^k$ is a length $+k$ S-T factorization with $k > 0$. ∎

The following is our main result.

Theorem 4.3.5 *Let G be a group and let L and R be nonempty subsets of G that consist of elements of finite order. Then $\Gamma = \mathrm{2SCay}(G; L, R)$ is strongly connected if and only if*

1. $G = \langle L\rangle\langle R\rangle$ *and*
2. *there exists an L^{-1}-R factorization $e = u_{L^{-1}}u_R$ of $e \in G$ such that $|\ell(u_{L^{-1}}) - \ell(u_R)| = 1$.*

Proof Assume $G = \langle L\rangle\langle R\rangle$ and, using Lemma 4.3.3 if necessary, that $e = u_{L^{-1}}u_R$ is a length $+1$ L^{-1}-R factorization of e. Fix $g \in G$. We will first show that there is a path from the identity element e of G to any element g of G.

Since $G = \langle L\rangle\langle R\rangle$, each $g \in G$ has some $\langle L\rangle$-$\langle R\rangle$ factorization and hence, by Lemma 4.3.3, an L^{-1}-R factorization $g = w_{L^{-1}}w_R$. If $\ell(w_{L^{-1}}) = \ell(w_R)$, then we write $g = w_{L^{-1}}ew_R$. If $\ell(w_{L^{-1}})-\ell(w_R) = k > 0$ then the equal-length L^{-1}-R factorization $g = (w_{L^{-1}}u^k_{L^{-1}})(u^k_R w_R)$ can be written as $g = (w_{L^{-1}}u^k_{L^{-1}})e(u^k_R w_R)$. If $\ell(w_{L^{-1}})-\ell(w_R) = k < 0$, first use Lemma 4.3.3 to obtain a length -1 L^{-1}-R factorization of e, insert such a factorization k times between the L^{-1} and R words to obtain an equal-length L^{-1}-R factorization of g, and then insert e. In all three cases we see there is a path from e to g in Γ.

We next show there is a path from g to e in Γ. Since $G = \langle L\rangle\langle R\rangle$, there is an $\langle L\rangle$-$\langle R\rangle$ factorization of g and hence, by Lemma 4.3.3, an L-R^{-1} factorization $g = w_L w_{R^{-1}}$. Writing $w_{L^{-1}}$ for w_L^{-1} and w_R for $w_{R^{-1}}^{-1}$ yields $e = w_{L^{-1}}gw_R$. If $\ell(w_{L^{-1}}) - \ell(w_R) = k$ for $k \in \mathbb{Z}$ (including $k = 0$), there is a length $+\,k$ L^{-1}-R factorization of e, say $e = u^k_{L^{-1}}u^k_R$ and then writing

$$e = u^k_{L^{-1}}u^k_R = u^k_{L^{-1}}eu^k_R = (u^k_{L^{-1}}w_{L^{-1}})g(w_R u^k_R)$$

provides a path from g to e in Γ. If $k = 0$, then there is a path from g to e. Since the arbitrary vertex g is strongly connected to e, Γ is strongly connected.

Conversely, suppose Γ is strongly connected. Then for each $g \in G$, there is a path from e to g and we have $g = w_{L^{-1}}ew_R = w_{L^{-1}}w_R$, proving that $G = \langle L\rangle\langle R\rangle$. Since $g = w_{L^{-1}}w_R$ is an equal-length L^{-1}-R factorization, taking $g = r^{-1}$ for $r \in R$ yields $e = w_{L^{-1}}(w_R r)$, a length $+\,1$ L^{-1}-R factorization of $e \in G$. ∎

When $2\mathrm{SCay}(G; L, R)$ is finite, for the sake of comparison we provide a more indirect proof of Theorem 4.3.5. This approach will use Proposition 2.2.9, that if the in-degree and out-degree at each vertex in a digraph are equal, then the digraph is weakly connected if and only if it is strongly connected.

Indirect proof of Theorem 4.3.5 If Γ is strongly connected, then, proceeding as in the first proof of Theorem 4.3.5, we conclude that $G = \langle L\rangle\langle R\rangle$ and there is a length $\pm\,1$ L^{-1}-R factorization of $e \in G$. Repeat the same argument as in that proof to conclude that if both these conditions hold then there is a path from e to each vertex g of Γ and hence Γ is weakly connected. Proposition 2.2.9 provides an alternative method to show that Γ is strongly connected.

We claim that for every vertex g of Γ, $\text{indeg}(g) = \text{outdeg}(g)$. For each pair (l^{-1}, r) in $L^{-1} \times R$, and for each vertex g of Γ, $g = l^{-1}(lgr^{-1})r$. Thus for each $(l^{-1}, r) \in L^{-1} \times R$ and each vertex g of Γ, there is an arc (lgr^{-1}, g) into vertex g, and these are all such arcs. Therefore, for all $g \in G$ $\text{indeg}(g) = \text{outdeg}(g)$ and, by Proposition 2.2.9, Γ is strongly connected. ∎

The following examples illustrate that both conditions of Theorem 4.3.5 are necessary for the two-sided Cayley digraph to be connected.

Example 4.10 Consider $\Gamma = \text{2SCay}(G; L, R)$ where $G = I_2(3)$, with $L = \{rs\}$ and $R = \{rs, r^2s\}$. Note that $\langle L\rangle\langle R\rangle = G$. All elements of L^{-1} and R are transpositions, therefore there is no length ± 1 L^{-1}-R factorization of e and hence Γ is disconnected. The graph Γ has the two components $\Gamma_e = \{e, r, r^2\}$ and $\Gamma_s = \{s, rs, r^2s\}$ as shown in Figure 4.4.

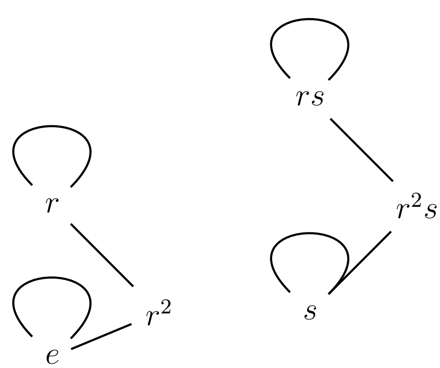

Figure 4.4: $\text{2SCay}(I_2(3); \{rs\}, \{rs, r^2s\})$

Example 4.11 Consider $I_2(3) = \langle r, s \mid r^3 = e, s^2 = e, rsrs = e\rangle$ and let $L = \{r^2s\}$, and $R = \{e, r^2s\}$. Then clearly $\langle L\rangle\langle R\rangle \neq I_2(3)$, but since $e \in R$, the second condition of Theorem 4.3.5 holds. The graph is disconnected with two components as shown in Figure 4.5.

Example 4.12 Consider the group $I_2(4) = \langle r, s \mid r^4 = e, s^2 = e, rsrs = e\rangle$,

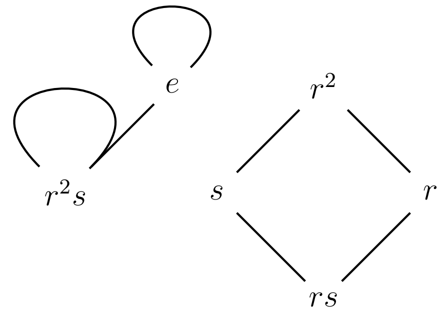

Figure 4.5: $2\mathrm{SCay}(I_2(3); \{r^2s\}, \{e, r^2s\})$

and let $L = \{r^3\}$ and $R = \{r^2s\}$. The corresponding graph, shown in Figure 4.6, is disconnected with two components. Note that $G = \langle L \rangle \langle R \rangle$. Hence there is no length ± 1 L^{-1}-R factorization of e. Otherwise the graph would be connected.

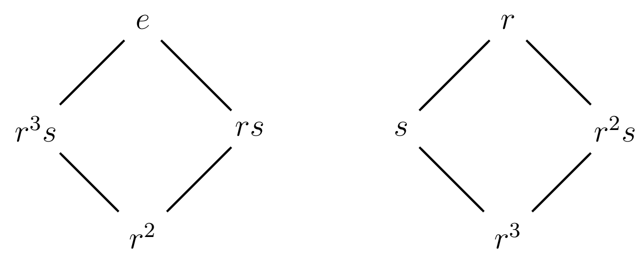

Figure 4.6: $2\mathrm{SCay}(I_2(4); \{r^3\}, \{r^2s\})$

The following result is an immediate consequence of Theorem 4.3.5 since the length ± 1 L^{-1}-R factorization of e is guaranteed by the containment of e in L or R.

Corollary 4.3.6 *Let G be a group and let L and R be nonempty subsets of G that consist of elements of finite order. If $\Gamma = 2\mathrm{SCay}(G; L, R)$ and $G = \langle L \rangle \langle R \rangle$ and $e \in L$ or $e \in R$, then Γ is connected.*

Proof Without loss of generality let $e \in L$. Every element of R has finite order. Let the order of $r \in R$ be n. Then, $e^{n+1}r^n = e$ is a length-1 L^{-1}-R factorization of e and the result follows by applying Theorem 4.3.5. ∎

We have considered a two-sided Cayley digraph $\Gamma = \mathrm{2SCay}(G; L, R)$ where G can be an infinite group and found conditions necessary for Γ to be connected provided that L and R consist of elements of finite order. The following result gives conditions for the connectedness of Γ where the elements in L or in R may have infinite order. Given any subset L of a group G, we will write $\langle L\rangle_{mon}$ to denote the monoid generated by L.

Theorem 4.3.7 *Let G be a group and L and R be nonempty subsets of G. Then $\mathrm{2SCay}(G; L, R)$ is connected if and only if*

1. *$G = \langle L^{-1}\rangle_{mon}\langle R\rangle_{mon}$ and $G = \langle L\rangle_{mon}\langle R^{-1}\rangle_{mon}$, and*
2. *there exist a length $+1$ and a length -1 L^{-1}-R factorization of e.*

Proof Let L and R be nonempty subsets of a group G and let $\mathrm{2SCay}(G; L, R)$ be connected. Then for each $g \in G$, there is a path from e to g and hence there is an L^{-1}-R factorization $g = w_{L^{-1}} e w_R = w_{L^{-1}} w_R$ where $\ell(w_{L^{-1}}) = \ell(w_R)$. Hence $g \in \langle L^{-1}\rangle_{mon}\langle R\rangle_{mon}$ and therefore $G = \langle L^{-1}\rangle_{mon}\langle R\rangle_{mon}$. Since the two-sided Cayley digraph $\mathrm{2SCay}(G; L, R)$ is strongly connected, there is also a path from g to e and hence $e = v_{L^{-1}} g v_R$ where $\ell(v_{L^{-1}}) = \ell(v_R)$. Solving for g gives $g = (v_L^{-1})^{-1} v_R^{-1}$ which implies that $g \in \langle L\rangle_{mon}\langle R^{-1}\rangle_{mon}$ and hence $G = \langle L\rangle_{mon}\langle R^{-1}\rangle_{mon}$. Since for each $g \in G$ there is a factorization $g = w_{L^{-1}} w_R$, choosing $g = l \in L$ gives $l = w_{L^{-1}} w_R$ and hence $e = (l^{-1} w_{L^{-1}}) w_R$ with $\ell(l^{-1} w_{L^{-1}}) = \ell(w_R) + 1$, while choosing $g = r^{-1}$ for $r \in R$ gives $\ell(w_R) = \ell(l^{-1} w_{L^{-1}}) - 1$ $r^{-1} = w_{L^{-1}} w_R$ and hence $e = w_{L^{-1}}(w_R r)$ with $\ell(w_{L^{-1}}) = \ell(w_R r) - 1$, hence $\ell(w_R r) = \ell(w_{L^{-1}}) + 1$.

Conversely, suppose conditions 1 and 2 hold. Then, arguing as in the proof of Theorem 4.3.5 yields that every $g \in G$ is strongly connected to e. ∎

4.3.2 Disconnected two-sided Cayley digraphs with three components

The statement of [7, Theorem 1.5] also includes, under the assumption $L = L^{-1}$ and $R = R^{-1}$, that if $G = \langle L\rangle\langle R\rangle$ but there is no L-R factorization of $e \in G$ into words whose lengths have opposite parity, then $\Gamma = 2\mathrm{SCay}(G; L, R)$ is disconnected with exactly two components. The cases when both conditions fail or when $G \neq \langle L\rangle\langle R\rangle$ but there is an opposite length parity L-R factorization of e are not addressed. In this subsection we treat the case where $G \neq \langle L\rangle\langle R\rangle$ and there is no length ± 1 L^{-1}-R factorization of e.

Proposition 4.3.8 *Let G be a group and let L and R be nonempty subsets of G that consist of elements of finite order. Let $\Gamma = 2\mathrm{SCay}(G; L, R)$. If*

1. *$G \neq \langle L\rangle\langle R\rangle$, and*

2. *there is no length ± 1 L^{-1}-R factorization of $e \in G$,*

then Γ is disconnected with at least three components.

Proof Since the conditions of Theorem 4.3.5 do not hold, Γ is disconnected. Since there is no length ± 1 L^{-1}-R factorization of e, L and R do not contain the identity element. Any element of G must be in $\langle L\rangle\langle R\rangle$ or in $G \setminus \langle L\rangle\langle R\rangle$, and both are nonempty since $e = e.e \in \langle L\rangle\langle R\rangle$ and $G \neq \langle L\rangle\langle R\rangle$. We find elements in three distinct components.

Element e is in some connected component, Γ_e, of Γ and is connected to all vertices g with an L^{-1}-R factorization $g = w_{L^{-1}}w_R$ such that $\ell(w_{L^{-1}}) = \ell(w_R)$.

Element $r^{-1} = er^{-1}$ is in $\langle L\rangle\langle R\rangle$ as well but is not in Γ_e. If r^{-1} were connected to e that would yield an equal-length L^{-1}-R factorization $r^{-1} = u_{L^{-1}}u_R$ and hence an L^{-1}-R factorization $e = u_{L^{-1}}(u_R r)$ with $\ell(u_R r) = \ell(u_{L^{-1}}) + 1$.

Since $G \setminus \langle L\rangle\langle R\rangle \neq \emptyset$, let $h \in G \setminus \langle L\rangle\langle R\rangle$. Then element h is not in Γ_e and not in $\Gamma_{r^{-1}}$ since if h were connected to e or r^{-1} then the connecting words in L^{-1} and R would yield an expression for h in $\langle L\rangle\langle R\rangle$.

Since elements e, r^{-1} and h are in distinct components of Γ this shows that Γ has at least three connected components Γ_e, $\Gamma_{r^{-1}}$, and Γ_h. ∎

In the following two examples $G \neq \langle L \rangle \langle R \rangle$ and there is no length ± 1 L^{-1}-R factorization of e. Both corresponding two-sided Cayley digraphs are disconnected with four and three components, illustrating the conclusions of Proposition 4.3.8.

Example 4.13 Consider the group $G = I_2(3)$, and subsets $L = \{rs\}$ and $R = \{r^2s\}$. Then $\langle L \rangle \langle R \rangle = \{e, rs, r^2s, r^2\} \neq G$ and, since both rs and r^2s have order 2, there is no length ± 1 L^{-1}-R factorization of $e \in G$. The graph $2\mathrm{SCay}(G; L, R)$ has the four components $\{e, r^2\}$, $\{s\}$, $\{r\}$ and $\{rs, r^2s\}$ as shown in Figure 4.7.

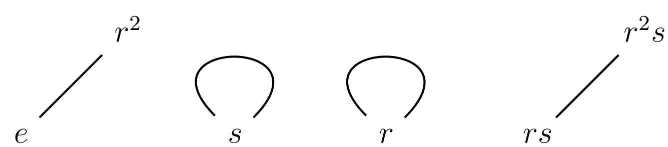

Figure 4.7: $2\mathrm{SCay}(I_2(3); \{rs\}, \{r^2s\})$

Example 4.14 Let $L = \{r^2s\}$ and $R = \{r^2, r^2s\}$ be subsets of the group $G = I_2(4) = \langle r, s \mid r^4 = e, s^2 = e, rsrs = e \rangle$. There is no length ± 1 L^{-1}-R factorization of $e \in G$ and $G \neq \langle L \rangle \langle R \rangle$. The corresponding two-sided Cayley digraph has the three components $\{e, s\}$, $\{r, r^3, rs, r^3s\}$ and $\{r^2, r^2s\}$. Figure 4.8 illustrates this example.

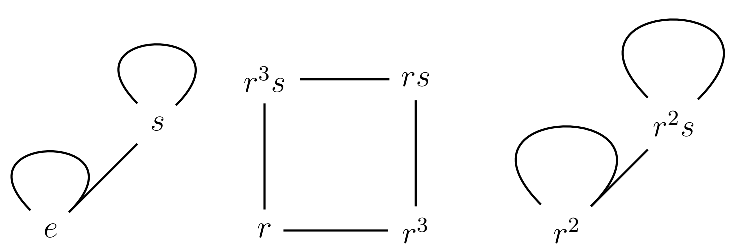

Figure 4.8: $2\mathrm{SCay}(I_2(3); \{r^2s\}, \{r^2, r^2s\})$

4.3.3 Disconnected two-sided Cayley graphs with two components

For inverse-closed subsets L and R of group G such that (L, R) has the $2S$-Cayley property, Theorem 1.5 of [7] proves that if $G = \langle L\rangle\langle R\rangle$ and if $\ell(w)$ and $\ell(w')$ have opposite parity for each L-R factorization $e = ww'$ of e, then the corresponding two-sided Cayley graph is disconnected with exactly two components. Dropping the $2S$-Cayley property from Theorem 1.5 does not affect connectedness, and we have the following result.

Proposition 4.3.9 *Let L and R be inverse-closed subsets of a group G. If $G = \langle L\rangle\langle R\rangle$ but the second condition of Theorem 4.3.1 does not hold, then $\Gamma = \mathrm{2SCay}(G; L, R)$ is disconnected with exactly two components.*

Proof Suppose that $G = \langle L\rangle\langle R\rangle$ and if $ww' = e$ is an L-R factorization of e then $\ell(w)$ and $\ell(w')$ have the same parity. Since $G = \langle L\rangle\langle R\rangle$, then each $g \in G$ has an L-R factorization of the form $g = w_L w_R$. If there are two factorizations $g = u_L u_R = v_L v_R$ so that in one factorization the lengths of the L and R words have the same parity while in the other factorization the lengths have opposite parity, then $v_L^{-1} u_L u_R v_R^{-1} = e$ is an L-R factorization of e with $\ell(v_L^{-1} u_L) = \ell(v_L^{-1}) + \ell(u_L)$ and $\ell(u_R v_R^{-1}) = \ell(u_R) + \ell(v_R)$ of opposite parity. This contradicts the hypothesis. This contradiction proves that the parity relationship between lengths of words in $L - R$ factorizations of g remain fixed as such words vary.

Case 1: Suppose $\ell(w_L)$ and $\ell(w_R)$ have the same parity.

- If $\ell(w_L) = \ell(w_R)$, then $g = w_L w_R = w_L e w_R$ and g is connected to e.
- If $\ell(w_L) \neq \ell(w_R)$, without loss of generality let $\ell(w_L) > \ell(w_R)$. Since $\ell(w_L) - \ell(w_R)$ is even, for any $r \in R$, right multiplying $g = w_L e w_R$ by $\frac{\ell(w_L)-\ell(w_R)}{2}$ copies of $rr^{-1} = e$ shows that g is connected to e.

Case 2: Suppose $\ell(w_L)$ and $\ell(w_R)$ have opposite parity. Then for any expression of g the lengths of the words in the L-R factorization have opposite parity and hence there is no path from e to g.

Now it will be shown that all elements of G with L-R factorizations where the lengths of the L and R words have opposite parity are connected. Let $g = w_L w_R$ with $\ell(w_L)$ and $\ell(w_R)$ of opposite parity and let $h = v_L v_R$ with $\ell(v_L), \ell(v_R)$ of opposite parity.

Note that

$$g = w_L v_L^{-1} h v_R^{-1} w_R.$$

Regardless of whether $\ell(w_L)$ and $\ell(v_L)$ are odd and $\ell(w_R)$ and $\ell(v_R)$ are even, or $\ell(w_R)$ and $\ell(v_R)$ are odd and $\ell(w_L)$ and $\ell(v_L)$ are even, or $\ell(w_L)$ and $\ell(v_L)$ are even and $\ell(w_R)$ and $\ell(v_R)$ are odd, $\ell(w_L v_L^{-1})$ and $\ell(v_R^{-1} w_R)$ have the same parity. Using $ll^{-1} = rr^{-1} = e$ as needed proves that g and h are connected. Therefore Γ is disconnected with exactly two connected components. ∎

We now consider the case where $G = \langle L \rangle \langle R \rangle$ but there is no length ± 1 L^{-1}-R factorization of $e \in G$. Under an additional condition that allows adjusting the relative lengths of the factorizations by two, Γ will consist of exactly two components.

Proposition 4.3.10 *Let G be a group and let L and R be nonempty subsets of G that consist of elements of finite order. Let $\Gamma = 2\mathrm{SCay}(G; L, R)$. Assume that*

1. *$G = \langle L \rangle \langle R \rangle$,*
2. *there is no length ± 1 L^{-1}-R factorization of $e \in G$, and*
3. *$L \cap L^{-1}$ or $R \cap R^{-1}$ is nonempty.*

Then Γ is disconnected with exactly two components.

Proof Since there is no length ± 1 L^{-1}-R factorization of $e \in G$ then Γ is disconnected. We identify the two components of Γ. Since $G = \langle L \rangle \langle R \rangle$, then every element $g \in G$ has an $\langle L \rangle$-$\langle R \rangle$ factorization. Hence, by Lemma 4.3.3, g

has an L^{-1}-R factorization $g = w_{L^{-1}}w_R$. Suppose that $L \cap L^{-1}$ is nonempty with $l, l^{-1} \in L^{-1}$.

If $\ell(w_{L^{-1}}) = \ell(w_R)$ then $g = w_{L^{-1}}ew_R$ is in Γ_e, the connected component of Γ containing e. If $\ell(w_R) - \ell(w_{L^{-1}}) = 2k > 0$ writing $g = [w_{L^{-1}}(l^{-1}l)^k]ew_R$ shows that g is in Γ_e. If $\ell(w_R) - (w_{L^{-1}}) = 2k + 1 > 0$ then $g = [w_{L^{-1}}(l^{-1}l)^k l^{-1}]lw_R$ shows that g is in Γ_l. Note that $e \notin \Gamma_l$ since otherwise there is a length-1 L^{-1}-R factorization of e.

If $\ell(w_{L^{-1}}) > \ell(w_R)$, then let $r \in R$ be of order m. Rewrite $g = w_{L^{-1}}w_R$ by inserting enough copies of r^m so that $g = w_{L^{-1}}w'_R$ with $\ell(w'_R) > \ell(w_{L^{-1}})$. Then proceed by the same argument to see that $g \in \Gamma_e$ or $g \in \Gamma_l$ and conclude that Γ has two components.

If $R \cap R^{-1}$ is nonempty, swap the roles of L and R above to yield $g \in \Gamma_e$ or $g \in \Gamma_r$, proving that Γ has two components in either case. ∎

Example 4.15 There exist two-sided Cayley digraphs for which the hypotheses of Proposition 4.3.10 hold. Consider the two-sided Cayley digraph Γ defined as $2\mathrm{SCay}(G; L, R)$ where $G = I_2(3)$, with $L = \{rs\}$ and $R = \{rs, r^2s\}$. Note that $\langle L\rangle\langle R\rangle = G$ and also $L \cap L^{-1} = \{rs\} \neq \emptyset$, and $R \cap R^{-1} = \{rs, r^2s\} \neq \emptyset$. Since all elements of L^{-1} and R are transpositions, then there is no length ± 1 L^{-1}-R factorization of e and hence Γ is disconnected. Observe that Γ has the two components $\Gamma_e = \{e, r, r^2\}$ and $\Gamma_s = \{s, rs, r^2s\}$. The graph of $\Gamma = (I_2(3); \{rs\}, \{rs, r^2s\})$ is shown in Figure 4.9.

The following example illustrates that the third condition of Proposition 4.3.10 is not a necessary condition. We give subsets L and R of the quaternion group $G = Q_8$ in which the first two conditions hold but the third fails, and yet the two-sided Cayley digraph $2\mathrm{SCay}(G; L, R)$ has two components.

Example 4.16 Let $G = Q_8$, the quaternion group, and $L = \{i, k\}$, and $R = \{-i, -k\}$. It can be verified that $2\mathrm{SCay}(G; L, R)$ has two components $\Gamma_e = \{\pm 1, \pm j\}$ and $\Gamma_i = \{\pm i, \pm k\}$. Note that the first two conditions of

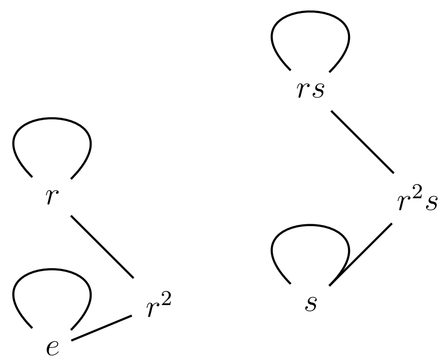

Figure 4.9: 2SCay$(I_2(3); \{rs\}, \{rs, r^2s\})$

Proposition 4.3.10 hold. However, $L^{-1} = \{-i, -k\}$ and $R^{-1} = \{i, k\}$ and hence $L \cap L^{-1} = \emptyset$ and $R \cap R^{-1} = \emptyset$. Therefore the third condition is not necessary for the two-sided Cayley digraph to have two components.

Let G be a group and let L and R be nonempty subsets of G that satisfy the hypotheses of Proposition 4.3.10. Then in fact the orders of elements of L and R are even, which can be used to construct an alternative proof of Proposition 4.3.10.

Proposition 4.3.11 *Let G be a group and let L and R be nonempty subsets of G that consist of elements of finite order. Let $\Gamma = \text{2SCay}(G; L, R)$. Assume that*

1. $G = \langle L \rangle \langle R \rangle$,
2. *there is no* length ± 1 L^{-1}-R *factorization of* $e \in G$, *and*
3. $L \cap L^{-1}$ *or* $R \cap R^{-1}$ *is nonempty.*

Then each element in L and each element in R has even order.

Proof Let G be a group and let L and R be nonempty subsets of G that consist of elements of finite order such that $G = \langle L \rangle \langle R \rangle$ and there is no length ± 1 $L^{-1} - R$ factorization of $e \in G$. Suppose also that $L \cap L^{-1} \neq \emptyset$. We will prove that the order of each $l \in L$ and each $r \in R$ must be even. Suppose, to the contrary, that there exists at least one element in L or in R whose order is odd.

Suppose $l_0 \in L$ has odd order $m = 2j+1$. Let $r \in R$ have order n. Take $l, l^{-1} \in L^{-1}$. If $n = 2k+1$ is odd, then $e = (ll^{-1})^k r^{2k+1}$ is a length $+\ 1$ L^{-1}-R factorization of e. If $n = 2k$ is an even number and $2j < 2k+1$, then $e = ((l_0^{-1})^{2j+1}(ll^{-1})^{k-j-1})r^{2k}$ is a length $+1$ L^{-1}-R factorization of e. If $n = 2k$ is even and $2j > 2k+1$, if necessary repeatedly multiply r^n by itself until $\ell((l_0^{-1})^m) < \ell(r^n \cdots r^n)$. Then, by the case above, there is a length $+1$ L^{-1}-R factorization of e.

Suppose that $r \in R$ has odd order $n = 2k+1$. Condition $L \cap L^{-1} \neq \emptyset$ implies that there exist l and l^{-1} in L^{-1}. Therefore $e = (ll^{-1})^k r^{2k+1}$ is a length $+\ 1$ L^{-1}-R factorization of e.

The case where $R \cap R^{-1} \neq \emptyset$ is treated in a similar way. We conclude that all elements of L and of R have even order. ∎

Remark 4.3.12 The results of Proposition 4.3.11 can be used to provide an alternative proof of Proposition 4.3.10. Suppose that L and R are subsets of group G that consist of elements of finite order. By Proposition 4.3.11, if element $l \in L$ and element $r \in R$ have orders m and n respectively, then both m and n are even. Hence, adjusting the length of any word by making use of $e = (l^{-1})^m$ or $e = r^n$ will not change the parity of the length of the word.

In Proposition 4.3.10 we proved that given nonempty subsets L and R of a group G such that $G = \langle L \rangle \langle R \rangle$ and either $L \cap L^{-1} \neq \emptyset$ or $R \cap R^{-1} \neq \emptyset$, if there is no length $\pm\ 1$ $L^{-1} - R$ factorization of $e \in G$ then Γ is disconnected with exactly two components. Observations made from computer generated examples show that for groups $G = I_2(3)$, $I_2(4)$ or $I_2(5)$ such that 2SCay$(G; L, R)$ has two components and $G = \langle L \rangle \langle R \rangle$, but the length $\pm\ 1$ condition fails, then $L \cap\ L^{-1} \neq \emptyset$ or $R \cap R^{-1} \neq \emptyset$. One question for further investigation is whether for any dihedral group if the corresponding two-sided Cayley graph has two components, the length±1 condition fails and $G = \langle L \rangle \langle R \rangle$, then either

$L \cap L^{-1} \neq \emptyset$ or $R \cap R^{-1} \neq \emptyset$.

While Theorem 4.3.5 gives conditions under which $\Gamma = 2\mathrm{SCay}(G; L, R)$ has only one component, Proposition 4.3.13 considers another extreme case, the conditions under which the number of components of Γ equals the size of the group G.

Proposition 4.3.13 *Let L and R be nonempty subsets of a group G. The number of components of the digraph $\Gamma = 2\mathrm{SCay}(G; L, R)$ equals the size of G if and only if $L = \{z\} = R$ for some $z \in Z(G)$.*

Proof Suppose that $L = \{z\} = R$ for some $z \in Z(G)$. Then for each $g \in G$, $z^{-1}gz = z^{-1}zg = g$. Hence $\{g\}$ is a component for each $g \in G$. Therefore the number of components equals the cardinality of G.

Conversely, suppose that the number of components equals the cardinality of G. Then for all $g \in G$, $\{g\}$ is a component and hence $l^{-1}gr = g$ for all $l \in L$ and $r \in R$. Taking $g = e$, gives $l = r$ for all $l \in L$ and $r \in R$ forcing L and R to consist of a single common element. Let $l = r = z$. Then $z^{-1}gz = g$ gives $zg = gz$, implying that $z \in Z(G)$ and $L = \{z\} = R$. ∎

More generally, when L and R are subsets of $Z(G)$, then $L^{-1}gR = L^{-1}Rg$ and hence by Theorem 4.2.3, $\Gamma = 2\mathrm{SCay}(G; L, R)$ is the Cayley digraph $\mathrm{Cay}(G, L^{-1}R)$ whose number of components is $[G : \langle L^{-1}R \rangle]$ by Proposition 3.2.3.

4.3.4 Other two-sided Cayley graphs with two components

We end by considering the situation where $G \neq \langle L \rangle \langle R \rangle$ and there is a length± 1 L^{-1}-R factorization of $e \in G$, and find some conditions on L and R under which Γ has exactly two connected components.

Motivated by the study of Cayley digraphs $\mathrm{Cay}(G, S)$ where the coset $\langle S \rangle g$ is the connected component of the graph containing $g \in G$, we consider how the component containing $g \in G$ in the two-sided Cayley digraph $2\mathrm{SCay}(G; L, R)$ compares to the double coset $\langle L \rangle g \langle R \rangle$.

Proposition 4.3.14 *Let L and R be nonempty subsets of a group G and let $\Gamma = 2\mathrm{SCay}(G; L, R)$. If Γ_g is the connected component of Γ containing vertex g, then $\Gamma_g \subset \langle L \rangle g \langle R \rangle$. If further L and R are subgroups of G, then $\Gamma_g = LgR$.*

Proof If $h \in \Gamma_g$, then h is strongly connected to g. Hence there exist equal-length words $w_{L^{-1}}$ and w_R in L^{-1} and R respectively such that $h = w_{L^{-1}} g w_R$. Therefore $\Gamma_g \subset \langle L \rangle g \langle R \rangle$.

If L and R are subgroups of G, then $L^{-1} = L = \langle L \rangle$ and $R = \langle R \rangle$. Let $h \in L^{-1} g R$. Then $h = l^{-1} g r$, where $l \in L$ and $r \in R$. As words in L^{-1} and R respectively, l^{-1} and both r have length one. Hence h is connected to g, that is $h \in \Gamma_g$. Therefore, $LgR \subset \Gamma_g$ and hence $\Gamma_g = LgR$. ∎

Remark 4.3.15 In general, $\Gamma_g \neq \langle L \rangle g \langle R \rangle$. Consider for example the group S_3 and subsets $L = \{(123)\} = R$. Then $\langle L \rangle = \{e, (123), (132)\} = \langle R \rangle$ and $\langle L \rangle e \langle R \rangle = \{e, (123), (132)\}$. However $\Gamma_e = \{e\}$ and hence $\Gamma_e \neq \langle L \rangle e \langle R \rangle$.

If L and R are subgroups of G, then $\langle L \rangle \langle R \rangle = LR$ and it is easy to see by Proposition 4.3.14 that $\Gamma = 2\mathrm{SCay}(G; L, R)$ is connected if and only if for each $g \in G$, $G = LgR$. Since $e \in L$ and $e \in R$ there is always a length ± 1 L^{-1}-R factorization of e so we find a condition under which Γ has exactly two components provided that $G \neq LR$.

Proposition 4.3.16 *Let L and R be subgroups of a group G, where $LR \neq G$. Then the digraph $\Gamma = 2\mathrm{SCay}(G; L, R)$ has exactly two components if and only if for all $g \in G \setminus LR$, then $LgR = G \setminus LR$.*

Proof Let L and R be subgroups of G. Then $\langle L \rangle \langle R \rangle = LR$. In addition, e is in L and in R. Hence there is a length ± 1 $L^{-1} - R$ factorization of e. Suppose that $LR \neq G$ and $\Gamma = 2\mathrm{SCay}(G; L, R)$ has exactly two components. One such component is Γ_e. Since L and R are subgroups, by Proposition 4.3.14, $\Gamma_e = LeR = LR$. Since Γ has exactly two components, the other component has to be $G \setminus \Gamma_e = G \setminus LR$. For any $g \in G \setminus LR$, we have $\Gamma_g \neq \Gamma_e$. By Proposition 4.3.14 $\Gamma_g = LgR$. Since Γ has exactly two components, then they have to be $\Gamma_e = LR$ and $\Gamma_g = LgR = G \setminus LR$.

Conversely, suppose that for all $g \in G \setminus LR$, $LgR = G \setminus LR$. Since L and R are subgroups of G, then $\Gamma_g = LgR = G \setminus LR$. Element $e = e.e \in LR$ is not connected to g. We have, $\Gamma_e = LeR = LR$. Since $\Gamma_e \cup \Gamma_g = G$, Γ_e and Γ_g are the only components of Γ. ∎

For L and R nonempty subsets of a dihedral group G, if $G \neq \langle L\rangle\langle R\rangle$ but the length ± 1 condition holds, then the following examples give some conditions under which $\Gamma = \text{2SCay}(G; L, R)$ is disconnected with exactly two components. These show that when G is a dihedral or an arbitrary group and $G \neq \langle L\rangle\langle R\rangle$, finding general conditions for Γ to have exactly two components may be subtle.

Example 4.17 Let $G = I_2(n) = \langle r, s \mid r^n = s^2 = rsrs = e\rangle$ and let L and R be nonempty subsets of $\langle r\rangle$. Suppose there exist $r^i \in L$ and $r^j \in R$ such that $\gcd(i-j, n) = 1$ and $\gcd(i+j, n) = 1$. Then $\Gamma = \text{2SCay}(G; L, R)$ has exactly two connected components.

Explanation: Since L and R are subsets of $\langle r\rangle$, then clearly $\langle L\rangle\langle R\rangle \neq I_2(n)$. Suppose $r^i \in L$ and $r^j \in R$ are such that $\gcd(i-j, n) = 1$ and $\gcd(i+j, n) = 1$. Then some elements of Γ_e are: e, $r^{-i}er^j = r^{-i+j}$, $r^{2(-i+j)}$, ..., $r^{(n-1)(-i+j)}$. These are n distinct elements since $i-j$ is coprime with n. Similarly, some elements of Γ_s are s, $r^{-i}sr^j = sr^{i+j}$, $sr^{2(i+j)}$, ..., $sr^{(n-1)(i+j)}$, which are n distinct elements since $i+j$ is coprime with n. Observe that the $2n$ elements that have been computed are all distinct and $|I_2(n)| = 2n$. Therefore, Γ has exactly the two connected components $\Gamma_e = \{e, r^{-i+j}, r^{2(-i+j)}, ..., r^{(n-1)(-i+j)}\}$ and $\Gamma_s = \{s, sr^{i+j}, sr^{2(i+j)}, ..., sr^{(n-1)(i+j)}\}$.

Example 4.18 Consider $G = I_2(n)$ with n even and let $L = \{r^2\}$ and $R = \{e, r^i s\}$ for $i \in \{0, ..., n-1\}$. Then $\Gamma = \text{2SCay}(G; L, R)$ has precisely two components. They are $\Gamma_e = \{e, r^2, r^4, ..., r^{n-2}, s, r^2s, r^4s, ..., r^{n-2}s\}$ and $\Gamma_r = \{r, r^3, ..., r^{n-1}, rs, r^3s, ..., r^{n-1}s\}$.

Example 4.19 In $G = I_2(3)$, all pairs of the form $L = \{r^is\}$, $R = \{e, r^js\}$ yield two components in $2\mathrm{SCay}(G; L, R)$. In $I_2(4)$ the pairs $L = \{r^is\}$ and $R = \{e, r^js\}$ give two components provided that $(i - j, 4) = 1$.

Example 4.20 Consider the dihedral group $G = I_2(3)$. Then each of the pairs L, R in the following table satisfies that $G \neq \langle L\rangle\langle R\rangle$ and there is a length ± 1 L^{-1}-R factorization of e. In each case $2\mathrm{SCay}(G; L, R)$ has two components.

L	R
$\{s\}$	$\{e, s\}$
$\{rs\}$	$\{e, rs\}$
$\{r^2s\}$	$\{e, r^2s\}$

Now consider $G = I_2(4)$ and the pairs of subsets L, R given in the following table. Then $G \neq \langle L\rangle\langle R\rangle$ and since $e \in R$, there is a length ± 1 L^{-1}-R factorization of e. The pairs of sets L, R appear to have the same structure as in $I_2(3)$. However for each of the pairs L, R, the graph $2\mathrm{SCay}(G; L, R)$ has more than two components.

L	R
$\{s\}$	$\{e, s\}$
$\{rs\}$	$\{e, rs\}$
$\{r^2s\}$	$\{e, r^2s\}$
$\{r^3s\}$	$\{e, r^3s\}$

The same situation holds for $I_2(5)$ with the following pairs of subsets.

L	$\{r^2s\}$
$\{s\}$	$\{r^3s\}$
$\{rs\}$	$\{r^4s\}$

R

$\{e, s\}$

$\{e, rs\}$

$\{e, r^2s\}$

$\{e, r^3s\}$

$\{e, r^4s\}$

Given these observations, an attempt was made, to no avail, to check if the orders of elements of $L^{-1}R$ may give a hint on the relations that give a graph with two components when $I_2(n) \neq \langle L\rangle\langle R\rangle$. We have yet to find general conditions on arbitrary nonempty subsets L and R of $I_2(n)$ that ensure that Γ is disconnected with exactly two components when $\langle L\rangle\langle R\rangle \neq I_2(n)$.

4.4 Questions for future inquiry

The following questions are yet to be addressed.

- Finding conditions under which $2\mathrm{SCay}(G; L, R)$ has two components when $G \neq \langle L\rangle\langle R\rangle$. A partial result was obtained for dihedral groups but a result for general groups has not been fully done. Starting with an attempt at finding a complete result for dihedral groups may be a good first step.

- Is it possible to count the number of components of a two-sided Cayley digraph if both conditions of the connectedness result fail? We only determined that if both conditions fail, then the digraph is disconnected into at least three components. We need conditions under which the digraph has exactly three, four, five, ..., etc, components.

- What are necessary and sufficient conditions for a two-sided Cayley digraph to be vertex-transitive (Iradmusa and Praeger)? Iradmusa and Praeger determined that if G is the product of the normalizers of L and R, then the corresponding two-sided Cayley digraph is vertex-transitive. Can necessary and sufficient conditions be obtained, maybe starting with a strengthening of this result?

- Necessary and sufficient conditions for a two-sided Cayley digraph to be edge-transitive.

- What other classes of two-sided Cayley digraphs are guaranteed to be Cayley digraphs? (Iradmusa and Praeger.) For certain factorizations of G, Iradmusa and Praeger determined that a two-sided Cayley digraph is a Cayley digraph. We need other classes of two-sided Cayley digraphs that are Cayley digraphs.

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